

RANDOM HOMOGENIZATION OF COERCIVE HAMILTON-JACOBI EQUATIONS IN 1D

HONGWEI GAO

ABSTRACT. In this paper, we will prove the random homogenization of general coercive non-convex Hamilton-Jacobi equations in one dimensional case. This extends the result of Armstrong, Tran and Yu when the Hamiltonian has a separable form $H(p, x, \omega) = H(p) + V(x, \omega)$ for any coercive $H(p)$.

1. INTRODUCTION

1.1. Overview. We study the Hamilton-Jacobi equation of the following form:

$$\begin{cases} u_t + H(Du, x, \omega) = 0 & (x, t) \in \mathbf{R}^d \times (0, +\infty) \\ u(x, 0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

The Hamiltonian $H(p, x, \omega)$ is stationary ergodic and $g(x) \in BUC(\mathbf{R}^d)$. The main issue in stochastic homogenization of Hamilton-Jacobi equation is to consider: for each $\epsilon > 0$, $\omega \in \Omega$, let $u^\epsilon(t, x, \omega)$ be the unique solution of the equation:

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}, \omega) = 0 & (x, t) \in \mathbf{R}^d \times (0, +\infty) \\ u^\epsilon(x, 0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

Prove that for a.e. $\omega \in \Omega$, as $\epsilon \rightarrow 0$, $u^\epsilon(t, x, \omega) \rightarrow \bar{u}(t, x)$ locally uniformly and $\bar{u}(t, x)$ is the unique solution of the homogenized equation:

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 & (x, t) \in \mathbf{R}^d \times (0, +\infty) \\ \bar{u}(x, 0) = g(x) & x \in \mathbf{R}^d \end{cases}$$

If $H(p, x, \omega)$ is convex with respect to $p \in \mathbf{R}^d$, stochastic homogenization was proved independently by Souganidis[9] and by Rezakhanlou-Tarver[7]. This result was extended to t-dependent case by Schwab [8] when the Hamiltonian has super-linear growth in p and by Jing-Souganidis-Tran[6] for Hamiltonians with the form $a(x, t, \omega)|p|$. For those quasi-convex Hamiltonians, Siconolfi and Davini [5] established the random homogenization in 1d, and the general dimensional case was proved by Armstrong-Souganidis[2].

It remains an open problem that whether random homogenization still holds if the Hamiltonian is non-convex. The first genuinely non-convex example of stochastic homogenization was provided by Armstrong-Tran-Yu[3] for a special class of Hamiltonians with the following typical form.

$$H(p, x, \omega) = (|p|^2 - 1)^2 + V(x, \omega), (p, x) \in \mathbf{R}^d \times \mathbf{R}^d$$

2010 *Mathematics Subject Classification.* 35B27.

Key words and phrases. stochastic homogenization, Hamilton-Jacobi equation, non-convex Hamiltonian, coercive.

Partially supported by DMS-1151919 .

In one dimensional case, the same author established in another paper[4] the random homogenization of separable Hamiltonians

$$H(p, x, \omega) = H(p) + V(x, \omega), (p, x) \in \mathbf{R} \times \mathbf{R} \text{ for any coercive } H(p)$$

Recently, Armstrong-Cardaliaguet[1] considered the homogenization of Hamiltonian $H(p, x, \omega)$ that is homogeneous in p and with the assumption of unit range of dependence on (x, ω) (basically, it means that $H(p, x, \omega)$ and $H(p, y, \omega)$ are independent once $|x - y| > 1$).

This paper is aimed to extend the result of Armstrong-Tran-Yu[4] to general coercive $H(p, x, \omega)$.

1.2. Assumption and main result. Consider the Hamiltonian $H(p, x, \omega)$ that is continuous in $(p, x) \in \mathbf{R} \times \mathbf{R}$ and measurable in $\omega \in \Omega$.

(A1) Stationary Ergodic: there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a group $\{\tau_y\}_{y \in \mathbf{R}}$ of \mathcal{F} -measurable, measure-preserving transformations $\tau_y : \Omega \rightarrow \Omega$, i.e. for any $x, y \in \mathbf{R}$:

$$\tau_{x+y} = \tau_x \circ \tau_y \text{ and } \mathbf{P}[\tau_y(A)] = \mathbf{P}[A]$$

Ergodic: $A \in \mathcal{F}$, $\tau_z(A) = A$ for every $z \in \mathbf{R} \Rightarrow \mathbf{P}[A] \in \{0, 1\}$.

Stationary: $H(p, y, \tau_z \omega) = H(p, y + z, \omega)$ for any $y, z \in \mathbf{R}$ and $\omega \in \Omega$.

(A2) Coercive: $\liminf_{|p| \rightarrow +\infty} \text{ess inf}_{(x, \omega) \in \mathbf{R} \times \Omega} H(p, x, \omega) = +\infty$.

(A3) Local Uniformly Continuous: for any compact set $K \subset \mathbf{R}$,

$$|H(p, x, \omega) - H(q, y, \omega)| \leq \rho_K(|p - q| + |x - y|), (p, x, \omega), (q, y, \omega) \in K \times \mathbf{R} \times \Omega$$

The above ρ_K is the modulus of continuity.

Theorem 1.1. Assume **(A1)-(A3)** hold and $g(x) \in BUC(\mathbf{R})$, for each $\epsilon > 0$ and $\omega \in \Omega$, let $u^\epsilon(x, t, \omega)$ be the solution of the Hamilton-Jacobi equation

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}, \omega) = 0 & (x, t) \in \mathbf{R} \times (0, +\infty) \\ u^\epsilon(x, 0) = g(x) & x \in \mathbf{R} \end{cases}$$

Then, there is an effective Hamiltonian $\overline{H}(p) \in C(\mathbf{R})$ with $\lim_{|p| \rightarrow +\infty} \overline{H}(p) = +\infty$, such that for a.e. $\omega \in \Omega$, $\lim_{\epsilon \rightarrow 0^+} u^\epsilon(x, t, \omega) = \overline{u}(x, t)$ locally uniformly and $\overline{u}(x, t)$ is the solution of the homogenized Hamilton-Jacobi equation

$$\begin{cases} \overline{u}_t + \overline{H}(D\overline{u}) = 0 & (x, t) \in \mathbf{R} \times (0, +\infty) \\ \overline{u}(x, 0) = g(x) & x \in \mathbf{R} \end{cases}$$

1.3. Main difficulty and main idea. Let's first review the case of separable Hamiltonian, by approximation, we can assume $H(p)$ has finite many wells. The main ingredients in the proof by Armstrong-Tran-Yu[4] are the following: (1) When the oscillation of $V(x, \omega)$ is larger than the maximal local oscillation of $H(p)$, $\overline{H}(p)$ turns out to be quasi-convex. (2) If $V(x, \omega)$ has small oscillation, they introduced gluing lemmas, through which the Hamiltonian can be eventually reduced to the large oscillation case.

For the general Hamiltonian $H(p, x, \omega)$, there are several difficulties we need to overcome.

First, unlike the separable Hamiltonian, the number of wells of $H(p, x, \omega)$ (as a function of p) depends on (x, ω) . To solve this problem, we approximate $H(p, x, \omega)$ by Hamiltonians that have same number of wells for every (x, ω) (c.f. section 3).

Secondly, we need to find a way to characterize the oscillation when p and (x, ω) are mixed. After that, we can extend the above (1) and (2) to our general situation.

2. PRELIMINARIES

2.1. Stability of Homogenization.

Definition 2.1. (Follow the definition in Armstrong-Tran-Yu[4]) $H(p, x, \omega)$ is regularly homogenizable at $p \in \mathbf{R}$ if there exists an $\overline{H}(p) \in \mathbf{R}$ such that: for any $\lambda > 0$, if $v_\lambda(x, p, \omega) \in W^{1,\infty}(\mathbf{R})$ is the unique viscosity solution of the equation

$$\lambda v_\lambda + H(v'_\lambda, x, \omega) = 0 \quad x \in \mathbf{R}$$

Then

$$(2.1) \quad \text{for any } R > 0, \quad \mathbf{P} \left[\omega \in \Omega : \limsup_{\lambda \rightarrow 0} \max_{|x| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p, \omega) + \overline{H}(p)| = 0 \right] = 1$$

Remark 2.2. By Armstrong-Souganidis[2], with **(A1)**, (2.1) is equivalent to the following identity.

$$\mathbf{P} \left[\omega \in \Omega : \lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p, \omega) + \overline{H}(p)| = 0 \right] = 1$$

Remark 2.3. Homogenization with $H(p, x, \omega)$ holds if $H(p, x, \omega)$ is regularly homogenizable for each $p \in \mathbf{R}$. If the cell problem at p is solvable, then $H(p, x, \omega)$ is regularly homogenizable at p .

Definition 2.4. Let $G(p, x, \omega) : \mathbf{R} \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ satisfy **(A1)**, denote

$$G_{\inf}(p) := \operatorname{ess\,inf}_{(x,\omega) \in \mathbf{R} \times \Omega} G(p, x, \omega) \quad G_{\sup}(p) := \operatorname{ess\,sup}_{(x,\omega) \in \mathbf{R} \times \Omega} G(p, x, \omega)$$

Lemma 2.5. If $G(p, x, \omega)$ satisfies **(A1)** and is continuous in x , $G_{\inf}(p), G_{\sup}(p) \in \mathbf{R}$, then for a.e. $\omega \in \Omega$,

$$G_{\inf}(p) = \operatorname{ess\,inf}_{x \in \mathbf{R}} G(p, x, \omega) \quad G_{\sup}(p) = \operatorname{ess\,sup}_{x \in \mathbf{R}} G(p, x, \omega)$$

Proof. Fix $p \in \mathbf{R}$, denote $g(x, \omega) = G(p, x, \omega)$. For any $\alpha \in \mathbf{R}$, define

$$(2.2) \quad A_\alpha := \{\omega \in \Omega : g(x, \omega) > \alpha \text{ for all } x \in \mathbf{R}\}$$

Stationary implies $\tau_z A_\alpha = A_\alpha$, for any $z \in \mathbf{R}$. By ergodicity, $\mathbf{P}[A_\alpha] = 0$ or 1. Let $\alpha_0 := \sup\{\alpha : \mathbf{P}[A_\alpha] = 1\}$. By (2.2), $\alpha_0 = G_{\inf}(p)$. Since $\mathbf{P} \left[A_{\alpha_0 - \frac{1}{n}} \right] = 1, n \in \mathbf{N}$, $\mathbf{P} \left[\bigcap_{n=1}^{\infty} A_{\alpha_0 - \frac{1}{n}} \right] = 1$.

Then

$$G_{\inf}(p) = \operatorname{ess\,inf}_{x \in \mathbf{R}} G(p, x, \omega) \quad \omega \in \bigcap_{n=1}^{\infty} A_{\alpha_0 - \frac{1}{n}}$$

The other equality can be proved similarly. □

Lemma 2.6. Given uniformly coercive Hamiltonians $\{H_n(p, x, \omega)\}_{n \geq 1} \cup \{H(p, x, \omega)\}$ that satisfy **(A1)**, each $H_n(p, x, \omega)$ has effective Hamiltonian $\overline{H}_n(p)$. And for a.e. $\omega \in \Omega$:

$$\lim_{n \rightarrow +\infty} \|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(K \times \mathbf{R})} = 0 \quad \text{for each compact set } K \subset \mathbf{R}$$

Then, $H(p, x, \omega)$ has effective Hamiltonian $\overline{H}(p)$ and $\lim_{n \rightarrow +\infty} \overline{H}_n(p) = \overline{H}(p)$.

Proof. Fix $p \in \mathbf{R}$, for $\lambda > 0$, let $v_{n,\lambda}(x, p, \omega)$ and $v_\lambda(p, x, \omega)$ be solutions of the following equations:

$$\lambda v_{n,\lambda} + H_n(p + v'_{n,\lambda}, x, \omega) = 0 \quad \lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0$$

Then

$$-\lambda v_{n,\lambda} \in [H_{n,\inf}(p), H_{n,\sup}(p)] \quad -\lambda v_\lambda \in [H_{\inf}(p), H_{\sup}(p)]$$

By uniform coercive, there is $r = r(p)$, such that $|v'_{n,\lambda}(x, \omega)|, |v'_\lambda(x, \omega)| < r$.

Denote $K := [p - r, p + r]$, then by comparison principle,

$$\begin{aligned} |\lambda v_{n,\lambda}(0, p, \omega) - \lambda v_\lambda(0, p, \omega)| &\leq \sup_{x \in \mathbf{R}} |\lambda v_{n,\lambda}(x, p, \omega) - \lambda v_\lambda(x, p, \omega)| \\ &\leq \|H_n(\cdot, \cdot, \omega) - H(\cdot, \cdot, \omega)\|_{L^\infty(K \times \mathbf{R})} \end{aligned}$$

Boundedness of $-\lambda v_{n,\lambda}$ implies $\{\overline{H_n}(p)\}_{n \geq 1}$ is bounded. For any subsequence $\{n_j\}_{j \geq 1}$, there is a sub-subsequence $\{n_{j_k}\}_{k \geq 1}$, such that $\lim_{k \rightarrow \infty} \overline{H_{n_{j_k}}}(p) = h_*$. Then

$$\begin{aligned} |(-\lambda v_\lambda(0, p, \omega)) - h_*| &\leq \left| (-\lambda v_\lambda(0, p, \omega)) - (-\lambda v_{n_{j_k},\lambda}(0, p, \omega)) \right| \\ &\quad + \left| (-\lambda v_{n_{j_k},\lambda}(0, p, \omega)) - \overline{H_{n_{j_k}}}(p) \right| + \left| \overline{H_{n_{j_k}}}(p) - h_* \right| \\ &\leq \|H_{n_{j_k}}(\cdot, \cdot, \omega) - H(\cdot, \cdot, \omega)\|_{L^\infty(K \times \mathbf{R})} \\ &\quad + \left| (-\lambda v_{n_{j_k},\lambda}(0, p, \omega)) - \overline{H_{n_{j_k}}}(p) \right| + \left| \overline{H_{n_{j_k}}}(p) - h_* \right| \\ &=: \textcircled{1} + \textcircled{2} + \textcircled{3} \end{aligned}$$

For any $\epsilon > 0$, when k is large enough, $\textcircled{1} < \frac{\epsilon}{3}$, and $\textcircled{3} < \frac{\epsilon}{3}$. Fix such k , there is some $\lambda_0 = \lambda_0(k)$, such that, $\textcircled{2} < \frac{\epsilon}{3}$ as long as $0 < \lambda < \lambda_0$. Thus $\lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) = h_*$, so $\overline{H}(p) = h_*$.

The above limit is independent of the choice of $\{n_j\}_{j \geq 1}$, then $\lim_{n \rightarrow \infty} \overline{H_n}(p) = h_*$. Thus

$$\lim_{n \rightarrow \infty} \overline{H_n}(p) = \overline{H}(p)$$

□

Remark 2.7. Based on this lemma, we can construct the approximation of $H(p, x, \omega)$ by constrained Hamiltonians(c.f. Definition 3.2), this is the first step of reduction in this paper.

Corollary 2.8. Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and fix $p_0 \in \mathbf{R}$.

(1) If $H(p, x, \omega)$ is regularly homogenizable on $(-\infty, p_0)$ and $\overline{H}(p)$ is continuous, then $H(p, x, \omega)$ is also homogenizable at p_0 and $\lim_{p \rightarrow p_0^-} \overline{H}(p) = \overline{H}(p_0)$.

(2) If $H(p, x, \omega)$ is regularly homogenizable on $(p_0, +\infty)$ and $\overline{H}(p)$ is continuous, then $H(p, x, \omega)$ is also homogenizable at p_0 and $\lim_{p \rightarrow p_0^+} \overline{H}(p) = \overline{H}(p_0)$.

Proof. Only prove (1), since the proof of (2) is similar. For any $\delta_n \rightarrow 0^+$, denote

$$H_n(p, x, \omega) := H(p - \delta_n, x, \omega)$$

By assumption, $H_n(p, x, \omega)$ is regularly homogenizable at p_0 . According to **(A3)**, for each $\omega \in \Omega$ and compact set $K \subset \mathbf{R}$, we have $\lim_{n \rightarrow \infty} \|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(K \times \mathbf{R})} = 0$.

Lemma 2.6 implies $H(p, x, \omega)$ is regularly homogenizable at p_0 and

$$\overline{H}(p_0) = \lim_{n \rightarrow +\infty} \overline{H}_n(p_0, x, \omega) = \lim_{n \rightarrow +\infty} \overline{H}(p_0 - \delta_n, x, \omega)$$

This is true for any sequence $\delta_n \rightarrow 0^+$, so

$$\lim_{p \rightarrow p_0^-} \overline{H}(p, x, \omega) = \overline{H}(p_0, x, \omega)$$

□

2.2. Comparison Principle.

Lemma 2.9. *Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)**, for $R > 0$, $1 \gg \lambda > 0$, $p \in \mathbf{R}$, let u and v both be viscosity solutions of the equation*

$$\lambda \gamma + H(p + \gamma', x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

If there is a constant $M = M(p) > 0$, such that $|\lambda u| + |\lambda v| \leq M(p)$. Then, there is a constant $C = C(p) > 0$, such that

$$|\lambda u - \lambda v| \leq \frac{M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)C(p)}{R} \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Proof. By $|\lambda u| + |\lambda v| \leq M(p)$, $H(p + u', x, \omega) \leq M(p)$, $H(p + v', x, \omega) \leq M(p)$.

By **(A2)**, there is some $r = r(p) > 0$, s.t. $|u'|, |v'| \leq r(p)$.

By **(A3)**, there is some $\delta = \delta(p) > 0$, s.t. $|H(q_1, x, \omega) - H(q_2, x, \omega)| < 1$ if

$$q_1, q_2 \in \left[p - r(p) - \frac{M(p)}{R}, p + r(p) + \frac{M(p)}{R}\right] \quad |q_1 - q_2| < \delta$$

Define $w(x) := v + \frac{M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)}{\delta(p)R\lambda}$, then $|w'| \leq r(p) + \frac{M(p)}{R}$.

Thus $H(p + w', x, \omega) \geq H(p + v', x, \omega) - \frac{M(p)}{\delta(p)R}$ and we have the following.

$$\begin{aligned} \lambda w + H(p + w', x, \omega) &= \lambda v + \frac{\lambda M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)}{\delta(p)R} + H(p + w', x, \omega) \\ &\geq \lambda v + \frac{\lambda M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)}{\delta(p)R} + H(p + v', x, \omega) - \frac{M(p)}{\delta(p)R} \\ &> 0 \end{aligned}$$

Moreover,

$$|\lambda u| + |\lambda v| \leq M(p) \implies w|_{|x|=\frac{R}{\lambda}} \geq v|_{|x|=\frac{R}{\lambda}} + \frac{M(p)}{\lambda} \geq u|_{|x|=\frac{R}{\lambda}}$$

By comparison principle, $w(x) \geq u(x)$, $x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$. So

$$u - v \leq \frac{M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)}{\delta(p)R\lambda}$$

Similarly,

$$v - u \leq \frac{M(p)}{R} \sqrt{|x|^2 + 1} + \frac{M(p)}{\delta(p)R\lambda}$$

Thus when $\lambda \leq 1$, let $C(p) = \frac{1}{\delta(p)}$, then for $x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$, we have

$$|\lambda u - \lambda v| \leq \frac{\lambda M(p) \sqrt{|x|^2 + 1}}{R} + \frac{C(p)M(p)}{R} \leq \frac{M(p) \sqrt{|x|^2 + 1}}{R} + \frac{C(p)M(p)}{R}$$

□

3. REDUCTION BY CONSTRAINED HAMILTONIAN WITH INDEX (\tilde{L}, L)

3.1. Approximation by cluster-point-free Hamiltonians. Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and denote

$$h_i^{(n)}(x, \omega) := H\left(\frac{i}{n}, x, \omega\right) \quad \mathcal{E}_n = \{h_i^{(n)}(x, \omega)\}_{-n^2 \leq i \leq n^2}$$

Let $\mathcal{E}_n^+ = \{g_i^{(n)}(x, \omega)\}_{-n^2 \leq i \leq n^2}$ be another family of stationary functions and define

$$\Delta_{\mathcal{E}_n, \mathcal{E}_n^+}(p, x, \omega) := \begin{cases} g_{-n^2}^{(n)} - h_{-n^2}^{(n)} & p \in (-\infty, -n) \\ (np - i) \left[g_{i+1}^{(n)} - h_{i+1}^{(n)} \right] + (i + 1 - np) \left[g_i^{(n)} - h_i^{(n)} \right] & p \in \left[\frac{i}{n}, \frac{i+1}{n} \right] \\ g_{n^2}^{(n)} - h_{n^2}^{(n)} & p \in (n, +\infty) \end{cases}$$

So $\Delta_{\mathcal{E}, \mathcal{E}^+}(p, x, \omega)$ is a stationary function and is continuous with respect to (p, x) .

Lemma 3.1. *If $H(p, x, \omega)$ satisfies **(A1)**-**(A3)**, then there is $\{H^{(n)}(p, x, \omega)\}_{n=2^j, j \in \mathbf{N}}$, such that*

*(1) $H^{(n)}(p, x, \omega)$ satisfies **(A1)**-**(A3)**, $\forall n = 2^j, j \in \mathbf{N}$.*

(2) For $-n^2 \leq i \leq n^2$, $H^{(n)}\left(\frac{i}{n}, x, \omega\right)$, as functions of x , have no cluster point.

(3) $\|H(p, x, \omega) - H^{(n)}(p, x, \omega)\|_{L^\infty(\mathbf{R} \times \mathbf{R} \times \Omega)} \leq \frac{1}{n}$.

Proof. For each $\epsilon > 0$, $-n^2 \leq i \leq n^2$, denote

$$H_\epsilon\left(\frac{i}{n}, x, \omega\right) := \frac{1}{\sqrt{2\pi\epsilon}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{2\epsilon}} H\left(\frac{i}{n}, y, \omega\right) dy$$

By **(A1)**, either $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has no cluster for a.e. $\omega \in \Omega$ or $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has cluster for a.e. $\omega \in \Omega$.

If $H_\epsilon\left(\frac{i}{n}, x, \omega\right)$ has a cluster point x_0 , without loss of generality, we can assume $x_0 = 0$ and $H_\epsilon\left(\frac{i}{n}, 0, \omega\right) = 0$. Then $\frac{\partial^k}{\partial x^k} H_\epsilon\left(\frac{i}{n}, 0, \omega\right) = 0, k = 0, 1, 2, \dots$, which means

$$\int_{\mathbf{R}} y^k e^{-\frac{y^2}{2\epsilon}} H\left(\frac{i}{n}, y, \omega\right) dy = 0$$

By Fourier analysis, we have $H\left(\frac{i}{n}, x, \omega\right) \equiv 0$. Denote

$$D_j := \left\{ \frac{i}{2^j} \right\}_{-4^j \leq i \leq 4^j} \quad D := \bigcup_{j=1}^{\infty} D_j$$

Then D is dense in \mathbf{R} , if $H_\epsilon(d, x, \omega)$ has a cluster point for all $d \in D$, then $H(d, x, \omega)$ is independent of (x, ω) for all $d \in D$. By continuity, $H(p, x, \omega)$ is independent of (x, ω) for all $p \in \mathbf{R}$, so it is already homogenized.

Thus, assume for some $d_0 \in D$, $H_\epsilon(d_0, x, \omega)$ has no cluster point. Since D_j is increasing, without loss of generality, assume $d_0 = 0 \in D_j, j \in \mathbf{N}$. For $j \in \mathbf{N}$ and $-4^j \leq i \leq 4^j$, define

$$g_i^{(2^j)}(x, \omega) := \begin{cases} H\left(\frac{i}{2^j}, x, \omega\right) & \text{if } H\left(\frac{i}{2^j}, x, \omega\right) \text{ has no cluster point} \\ H\left(\frac{i}{2^j}, x, \omega\right) + \frac{1}{2^j} \frac{H_\epsilon(0, x, \omega)}{\|H_\epsilon(0, x, \omega)\|_{L^\infty(\mathbf{R} \times \Omega)} + 1} & \text{if } H\left(\frac{i}{2^j}, x, \omega\right) \text{ has a cluster point} \end{cases}$$

Denote

$$\mathcal{E}_{2^j} := \{h_i^{(2^j)}(x, \omega)\}_{-4^j \leq i \leq 4^j} \quad \mathcal{E}_{2^j}^+ := \{g_i^{(2^j)}(x, \omega)\}_{-4^j \leq i \leq 4^j}$$

We can finish the proof by defining

$$H^{(n)}(p, x, \omega) := H(p, x, \omega) + \Delta_{\varepsilon_{2^j}, \varepsilon_{2^j}^+}(p, x, \omega) \quad n = 2^j$$

□

3.2. Approximation by constrained Hamiltonians. In this subsection, we find a way to approximate $H(p, x, \omega)$ by $\{H_n(p, x, \omega)\}_{n \geq 1}$ in the sense of Lemma 2.6. Here each $H_n(p, x, \omega)$ is constrained in the following sense.

Definition 3.2 (Constrained Hamiltonian). A Hamiltonian $H(p, x, \omega)$ is called constrained if it satisfies the following (1)-(5).

- (1) There is $k \in \mathbf{N}$ and $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_{k-1} < b_{k-1} < a_k < +\infty$.
- (2) For each (x, ω) , $H(p, x, \omega)|_{(-\infty, a_1)}$, $H(p, x, \omega)|_{(b_1, a_2)}$, \cdots , $H(p, x, \omega)|_{(b_{k-1}, a_k)}$ are decreasing.
- (3) For each (x, ω) , $H(p, x, \omega)|_{(a_k, +\infty)}$, $H(p, x, \omega)|_{(a_{k-1}, b_{k-1})}$, \cdots , $H(p, x, \omega)|_{(a_1, b_1)}$ are increasing.
- (4) $H(p, x, \omega)$ is Lipschitz with respect to p (with Lipschitz constant \mathcal{L}), uniformly in $(x, \omega) \in \mathbf{R}$.
- (5) Each of $H(a_i, x, \omega)$, $H(b_j, x, \omega)$, $1 \leq i \leq k$, $1 \leq j \leq k-1$ has no cluster point.

Lemma 3.3. If $H(p, x, \omega)$ satisfies **(A1)**-(**A3**), then for $n = 2^j$, $j \in \mathbf{N}$, there is $H_n(p, x, \omega)$, such that

- (a) $\{H(p, x, \omega)\}_{n \geq 1}$ is uniformly coercive.
- (b) Each $H_n(p, x, \omega)$ satisfies **(A1)**-(**A3**).
- (c) Each $H_n(p, x, \omega)$ is constrained.
- (d) Fix any $\delta > 0$. Then for any compact set $K \subset \mathbf{R}$, there is an $N \in \mathbf{N}$,

$$\|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(K \times \mathbf{R} \times \Omega)} < \delta \quad \text{if } n > N$$

Proof. According to Lemma 3.1, without loss of generality, we can assume each of $H(\frac{i}{n}, x, \omega)$, $-n^2 \leq i \leq n^2$ has no cluster point. We construct $H_n(p, x, \omega)$ by the following procedure.

STEP 1: For each $p \in (-\infty, n) \cup (n, \infty)$, define

$$H_n(p, x, \omega) = \begin{cases} |p+n| + H(-n, x, \omega) & p \in (-\infty, -n) \\ |p-n| + H(n, x, \omega) & p \in (n, +\infty) \end{cases}$$

STEP 2: For $k = 0, 1, 2, \dots, 2n^2$, define

$$H_n\left(-n + \frac{k}{n}, x, \omega\right) = H\left(-n + \frac{k}{n}, x, \omega\right)$$

STEP 3: For $i = 0, 1, 2, \dots, 2n^2 - 1$, define

$$H_n\left(-n + \frac{i}{n} + \frac{1}{2n}, x, \omega\right) = \max\left\{H\left(-n + \frac{i}{n}, x, \omega\right), H\left(-n + \frac{i+1}{n}, x, \omega\right)\right\} + \frac{1}{n}$$

STEP 4: For $i = 0, 1, 2, \dots, 2n^2 - 1$,

- (1) If $p \in \left(-n + \frac{i}{n}, -n + \frac{i}{n} + \frac{1}{2n}\right)$, then there is some $\theta \in (0, 1)$, such that

$$p = \theta \times \left(-n + \frac{i}{n}\right) + (1 - \theta) \times \left(-n + \frac{i}{n} + \frac{1}{2n}\right)$$

Then we define

$$H_n(p, x, \omega) = \theta H\left(-n + \frac{i}{n}, x, \omega\right) + (1 - \theta) H\left(-n + \frac{i}{n} + \frac{1}{2n}, x, \omega\right)$$

(2) If $p \in \left(-n + \frac{i}{n} + \frac{1}{2n}, -n + \frac{i+1}{n}\right)$, then there is some $\theta \in (0, 1)$, such that

$$p = \theta \times \left(-n + \frac{i}{n} + \frac{1}{2n}\right) + (1 - \theta) \times \left(-n + \frac{i+1}{n}\right)$$

Then we define

$$H_n(p, x, \omega) = \theta H\left(-n + \frac{i}{n} + \frac{1}{2n}, x, \omega\right) + (1 - \theta) H\left(-n + \frac{i+1}{n}, x, \omega\right)$$

(a) Since $H(p, x, \omega)$ satisfies **(A2)**, $\{H_n(p, x, \omega)\}_{n \geq 1}$ is uniformly coercive.

(b) By **(A1)**, $H\left(-n + \frac{k}{n}, x, \omega\right)$ is stationary, for $k = 0, 1, 2, \dots, 2n^2$. So, $H_n(p, x, \omega)$, as a linear combination of these functions, is stationary and satisfies **(A1)**-**(A3)**.

(c) By the above construction, such $H_n(p, x, \omega)$ is constrained with $2n^2+1$ wells. And $H_n(p, x, \omega)$ has Lipschitz constant $\mathcal{L} = 1 + n\rho_{[-n^2, n^2]}(\frac{1}{n})$ in p variable, uniformly in $(x, \omega) \in \mathbf{R}$.

(d) By **(A3)**, there is $N \in \mathbf{N}$, such that $N > \frac{3}{\delta}$, $K \subset [-N, N]$ and

$$p, q \in K, |p - q| < \frac{1}{N} \implies |H(p, x, \omega) - H(q, x, \omega)| < \frac{\delta}{3}$$

To prove (d), it suffices to show that: fix any $k \in \{0, 1, 2, \dots, 2n^2 - 1\}$

$$\|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty((K \cap (-n + \frac{k}{n}, -n + \frac{k+1}{n})) \times \mathbf{R})} < \delta$$

Denote

$$p_1 = -n + \frac{k}{n}, \quad p_2 = -n + \frac{k}{n} + \frac{1}{2n}, \quad p_3 = -n + \frac{k+1}{n}$$

Without loss of generality, assume that

$$H(p_1, x, \omega) \leq H(p_3, x, \omega)$$

Case 1: $p \in K \cap (p_1, p_2)$. Then there is some $\theta \in (0, 1)$ such that $p = \theta p_1 + (1 - \theta)p_2$,

$$\begin{aligned} |H_n(p, x, \omega) - H(p, x, \omega)| &= |H_n(\theta p_1 + (1 - \theta)p_2, x, \omega) - H(\theta p_1 + (1 - \theta)p_2, x, \omega)| \\ &= \left| \theta H(p_1, x, \omega) + (1 - \theta) \left[H(p_3, x, \omega) + \frac{1}{n} \right] \right. \\ &\quad \left. - H(\theta p_1 + (1 - \theta)p_2, x, \omega) \right| \\ &\leq \theta |H(p_1, x, \omega) - H(\theta p_1 + (1 - \theta)p_2, x, \omega)| \\ &\quad + (1 - \theta) |H(p_3, x, \omega) - H(\theta p_1 + (1 - \theta)p_2, x, \omega)| + \frac{1 - \theta}{n} \\ &< \delta \end{aligned}$$

Case 2: $p \in K \cap (p_2, p_3)$. Then there is some $\theta \in (0, 1)$ such that $p = \theta p_2 + (1 - \theta)p_3$,

$$\begin{aligned} |H_n(p, x, \omega) - H(p, x, \omega)| &= |H_n(\theta p_2 + (1 - \theta)p_3, x, \omega) - H(\theta p_2 + (1 - \theta)p_3, x, \omega)| \\ &= \left| \theta \left[H(p_3, x, \omega) + \frac{1}{n} \right] + (1 - \theta) H(p_3, x, \omega) \right. \\ &\quad \left. - H(\theta p_2 + (1 - \theta)p_3, x, \omega) \right| \\ &\leq |H(p_3, x, \omega) - H(\theta p_2 + (1 - \theta)p_3, x, \omega)| + \frac{\theta}{n} \\ &< \delta \end{aligned}$$

The above is true for all $k = 0, 1, 2, \dots, 2n^2 - 1$, thus

$$\|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(K \times \mathbf{R} \times \mathbf{R})} < \delta$$

□

Remark 3.4. By Lemma 2.6 and Lemma 3.3, to prove Theorem 1.1, it suffices to consider such Hamiltonian $H(p, x, \omega)$ that is constrained (3.2) and satisfies **(A1)**-(**A3**). So in the following sections, we only consider constrained Hamiltonians.

3.3. Constrained Hamiltonian with index (\tilde{L}, L) .

Definition 3.5. $H(p, x, \omega)$ is called constrained Hamiltonian with index (\tilde{L}, L) if

- (1) $H(p, x, \omega)$ is constrained (3.2).
- (2) $(a_1, b_1, a_2, b_2, \dots, a_{k-1}, b_{k-1}, a_k) = (\tilde{p}_1, \tilde{q}_1, \tilde{p}_2, \tilde{q}_2, \dots, \tilde{p}_{\tilde{L}}, \tilde{q}_{\tilde{L}}, 0, q_L, p_L, q_{L-1}, p_{L-1}, \dots, q_1, p_1)$.
- (3) $\text{ess sup}_{(x, \omega)} H(\tilde{p}_i, x, \omega) > 0, 1 \leq i \leq \tilde{L}$; $\text{ess sup}_{(x, \omega)} H(0, x, \omega) = 0$; $\text{ess sup}_{(x, \omega)} H(p_j, x, \omega) > 0, 1 \leq j \leq L$.
- (4) Each of $H(a_i, x, \omega), H(b_i, x, \omega), 1 \leq i \leq k$ has no cluster point.

Remark 3.6. Apply perturbation and shift coordinates if necessary, it suffices to consider homogenization of any constrained Hamiltonian with index (\tilde{L}, L) . The following example is a constrained Hamiltonian with index $(1, 2)$.

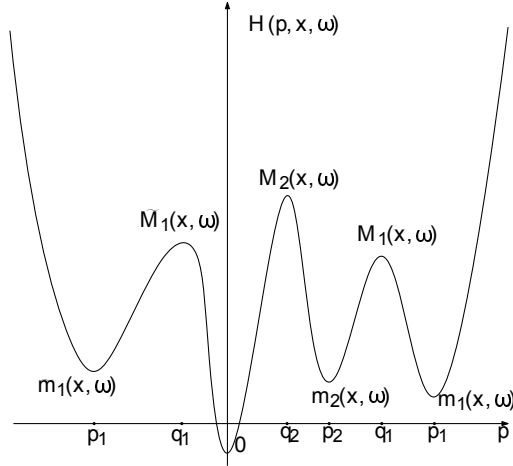


FIGURE 1. Constrained Hamiltonian with index $(1, 2)$

Notation 3.7. Let $H(p, x, \omega)$ be a constrained Hamiltonian with index (\tilde{L}, L) .

- (1) For each (x, ω) , denote monotone branches of $H(p, x, \omega)$ by

$$H|_{[p_1, \infty)} := \phi_{1, (x, \omega)}(p), \quad H|_{[q_1, p_1]} := \phi_{2, (x, \omega)}(p), \quad \dots, \quad H|_{[0, q_L]} := \phi_{2L+1, (x, \omega)}(p)$$

$$H|_{[-\infty, \tilde{p}_1]} := \tilde{\phi}_{1, (x, \omega)}(p), \quad H|_{[\tilde{p}_1, \tilde{q}_1]} := \tilde{\phi}_{2, (x, \omega)}(p), \quad \dots, \quad H|_{[\tilde{q}_{\tilde{L}}, 0]} := \tilde{\phi}_{2\tilde{L}+1, (x, \omega)}(p)$$

- (2) Denote inverse function of each branch by

$$(\phi_{i, (x, \omega)}(\cdot))^{-1} := \psi_{i, (x, \omega)}(\cdot) \quad \left(\tilde{\phi}_{i, (x, \omega)}(\cdot) \right)^{-1} := \tilde{\psi}_{i, (x, \omega)}(\cdot)$$

- (3) Denote local extreme values by

$$m_i(x, \omega) := H(p_i, x, \omega) \quad \tilde{m}_i(x, \omega) := H(\tilde{p}_i, x, \omega)$$

$$M_i(x, \omega) := H(q_i, x, \omega) \quad \tilde{M}_i(x, \omega) := H(\tilde{q}_i, x, \omega)$$

(4) Define two functions

$$(3.1) \quad m(x, \omega) := \min \left\{ \min_{1 \leq i \leq L} m_i(x, \omega), \min_{1 \leq j \leq \tilde{L}} \tilde{m}_j(x, \omega) \right\}$$

$$(3.2) \quad M(x, \omega) := \max \left\{ \max_{1 \leq i \leq L} M_i(x, \omega), \max_{1 \leq j \leq \tilde{L}} \tilde{M}_j(x, \omega) \right\}$$

4. AUXILIARY LEMMAS FOR GLUING LEMMAS

4.1. Estimation of Gradient.

Lemma 4.1. *Let Hamiltonian $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and be regularly homogenizable at p_0 , for each $\lambda > 0$, let $v_\lambda(x, p_0, \omega)$ be the viscosity solution of the equation:*

$$\lambda v_\lambda + H(p_0 + v'_\lambda, x, \omega) = 0$$

fix $P \in \mathbf{R}$, denote $\underline{P} := \operatorname{ess\,inf}_{(x, \omega)} H(P, x, \omega)$ and $\overline{P} := \operatorname{ess\,sup}_{(x, \omega)} H(P, x, \omega)$, then, there is an $\tilde{\Omega} \subset \Omega$

with $\mathbf{P}[\tilde{\Omega}] = 1$, such that, for each $\omega \in \tilde{\Omega}$, the following hold.

(1) If $\overline{H}(p_0) < \underline{P}$, $p_0 < P$, then for any $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x, p_0, \omega) \leq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(2) If $\overline{H}(p_0) < \underline{P}$, $p_0 > P$, then for any $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x, p_0, \omega) \geq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(3) If $\overline{H}(p_0) > \overline{P}$, $p_0 < P$, then for any $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x, p_0, \omega) \leq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(4) If $\overline{H}(p_0) > \overline{P}$, $p_0 > P$, then for any $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x, p_0, \omega) \geq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Proof of periodic case. (1) For p_0 , we have the cell problem

$$H(p_0 + v', x) = \overline{H}(p_0)$$

Suppose (1) is not true, then there is $x_1 \in [0, 1]$, such that $p_0 + v'(x_1) > P$. On the other hand,

$$\int_1^2 p_0 + v'(x) - P dx = p_0 - P < 0$$

So there is some $y_1 \in [1, 2]$, such that $p_0 + v'(y_1) - P < 0$. Then $\Psi(x) := p_0 + v(x) - Px$ attains local maximum at some $z_1 \in [x_1, y_1]$, so

$$\underline{P} \leq H(P, z_1) \leq \overline{H}(p_0) < \underline{P}$$

This is a contradiction, so we proved (1). The proofs of (2)(3)(4) are similar. \square

Proof of random case. (1) If it is not true, then there is $\Omega_1 \subset \Omega$, $\mathbf{P}[\Omega_1] > 0$, for any $\omega \in \Omega_1$, there are $R_1 = R_1(p_0, \omega) > 0$ and $\lambda_n \rightarrow 0$ such that

$$p_0 + v'_{\lambda_n}(x_{\lambda_n}, p_0, \omega) > P \quad \text{for some } x_{\lambda_n} \in \left[-\frac{R_1}{\lambda_n}, \frac{R_1}{\lambda_n}\right]$$

Denote $\delta := P - p_0 > 0$. For any $R > 0$, we have

$$\left| \frac{\lambda}{R} \int_{\frac{R_1}{\lambda}}^{\frac{R+R_1}{\lambda}} v'_\lambda(s, \omega) ds \right| \leq \frac{2(H_{\sup}(p_0) - H_{\inf}(p_0))}{R}$$

Fix any $R_2 = R_2(p_0) > \frac{4(H_{\sup}(p_0) - H_{\inf}(p_0))}{\delta}$, thus for any $R \geq R_2$, we have

$$\left| \frac{\lambda}{R} \int_{\frac{R_1}{\lambda}}^{\frac{R+R_1}{\lambda}} v'_\lambda(s, p_0, \omega) ds \right| < \frac{\delta}{2} \text{ for any } \lambda > 0$$

So

$$\frac{\lambda_n}{R_2} \int_{\frac{R_1}{\lambda_n}}^{\frac{R_2+R_1}{\lambda_n}} p_0 + v'_{\lambda_n}(s, p_0, \omega) - P ds \leq p_0 - P + \frac{\delta}{2} < 0$$

This implies

$$p_0 + v'_{\lambda_n}(y_{\lambda_n}, \omega) - P < 0 \quad \text{for some } y_{\lambda_n} \in \left(\frac{R_1}{\lambda_n}, \frac{R_2+R_1}{\lambda_n} \right)$$

Denote $\Psi(x, \omega) = p_0 x + v_{\lambda_n}(x, \omega) - P x$, then

$\Psi(x, \omega)$ is increasing (decreasing) in a neighborhood of $x_{\lambda_n}(y_{\lambda_n})$

Since $x_{\lambda_n} < y_{\lambda_n}$, $\Psi(x, \omega)$ attains local maximum at some $z_{\lambda_n} \in (x_{\lambda_n}, y_{\lambda_n})$. So

$$(4.1) \quad \lambda_n v_{\lambda_n}(z_{\lambda_n}, \omega) + H(P, z_{\lambda_n}, \omega) \leq 0$$

Since $H(p, x, \omega)$ is regularly homogenizable at p_0 , there is $\Omega_2 \subset \Omega$, s.t. $\mathbf{P}[\Omega_2] = 1$,

$$\limsup_{\lambda \rightarrow 0} \sup_{|x| \leq \frac{R_1+R_2}{\lambda}} |\lambda v_\lambda(x, \omega) + \overline{H}(p_0)| = 0 \text{ for each } \omega \in \Omega_2$$

Denote $\tau := \underline{P} - \overline{H}(p_0) > 0$, $\hat{\Omega} := \Omega_1 \cap \Omega_2$. So there is some $N_1(\omega)$,

$$(4.2) \quad \sup_{|x| \leq \frac{R_1+R_2}{\lambda_n}} |\lambda_n v_{\lambda_n} + \overline{H}(p_0)| < \frac{\tau}{2} \text{ for any } n \geq N_1$$

$$\mathbf{P}[\Omega_1] > 0, \mathbf{P}[\Omega_2] = 1 \implies \mathbf{P}[\hat{\Omega}] > 0 \implies \hat{\Omega} \neq \emptyset$$

Choose any $\omega \in \hat{\Omega}$ and $n \geq N_1(\omega)$, by (4.1) and (4.2),

$$\underline{P} \leq H(P, z_{\lambda_n}, \omega) \leq -\lambda_n v_{\lambda_n}(z_{\lambda_n}, \omega) \leq \overline{H}(p_0) + \frac{\tau}{2} = \underline{P} - \tau + \frac{\tau}{2} = \underline{P} - \frac{\tau}{2}$$

This is a contradiction. Thus (1) is proved. The proofs of (2)(3)(4) are similar. \square

Lemma 4.2. *Let Hamiltonian $H(p, x, \omega)$ satisfy **(A1)-(A3)** and be regularly homogenizable at $p_0 \in \mathbf{R}$ to $\overline{H}(p_0)$, for each λ , let $v_\lambda(x)$ be the viscosity solution of the following equation:*

$$\lambda v_\lambda(x) + H(p_0 + v'_\lambda(x), x, \omega) = 0$$

For $P, Q \in \mathbf{R}$, denote

$$\begin{aligned} \underline{P} &:= \operatorname{ess\,inf}_{(x, \omega)} H(P, x, \omega) & \overline{P} &:= \operatorname{ess\,sup}_{(x, \omega)} H(P, x, \omega) \\ \underline{Q} &:= \operatorname{ess\,inf}_{(x, \omega)} H(Q, x, \omega) & \overline{Q} &:= \operatorname{ess\,sup}_{(x, \omega)} H(Q, x, \omega) \end{aligned}$$

Then, there is an $\tilde{\Omega} \subset \Omega$ with $\mathbf{P}[\tilde{\Omega}] = 1$, such that for each $\omega \in \tilde{\Omega}$, the following hold.

(1) If $p_0 < P$, $P < Q$ and $\overline{P} < \underline{Q}$, then for each $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega)$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(2) If $p_0 < P$, $P < Q$ and $\underline{P} > \overline{Q}$, then for each $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega)$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(3) If $p_0 > P$, $P > Q$ and $\overline{P} < \underline{Q}$, then for each $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega)$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x) \geq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

(4) If $p_0 > P$, $P > Q$ and $\underline{P} > \overline{Q}$, then for each $R > 0$, there is $\lambda_0 = \lambda_0(R, p_0, \omega)$,

$$0 < \lambda < \lambda_0 \implies p_0 + v'_\lambda(x) \geq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Proof. (1) Case 1: $\overline{H}(p_0) < \underline{P}$, apply (1) of Lemma 4.1 to (p_0, P) .

Case 2: $\overline{H}(p_0) > \overline{P}$, apply (3) of Lemma 4.1 to (p_0, P) .

Case 3: $\overline{H}(p_0) \in [\underline{P}, \overline{P}]$, apply (1) of Lemma 4.1 to (p_0, Q) .

The proofs of (2)(3)(4) are similar. □

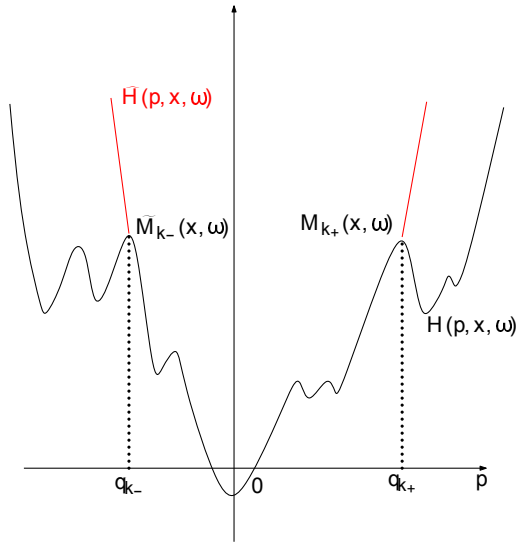


FIGURE 2. Squeeze

4.2. Squeeze Lemma.

Lemma 4.3. Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and be constrained with index (\tilde{L}, L) , if $H(p, x, \omega)$ has effective Hamiltonian $\overline{H}(p)$ with $\overline{H}(q) = 0$, then the following are true.

(1) If $q > 0$ and $\overline{H}|_{(q, +\infty)} > 0$, then $\overline{H}(p) \equiv 0$ for all $p \in [0, q]$.

(2) If $q < 0$ and $\overline{H}|_{(-\infty, q)} > 0$, then $\overline{H}(p) \equiv 0$ for all $p \in [q, 0]$.

Proof. (1) Recall the notation (3.1) and $H(p, x, \omega)$ is constrained with index (\tilde{L}, L) , we have

$$(4.3) \quad \mathbf{E}[m(x, \omega) > 0] > 0$$

Denote:

$$\begin{aligned} \underline{M}_i &:= \operatorname{ess\,inf}_{(x, \omega) \in \mathbf{R} \times \Omega} M_i(x, \omega) & \underline{M}^+ &:= \max_{1 \leq i \leq L} \underline{M}_i \\ \widetilde{\underline{M}}_i &:= \operatorname{ess\,inf}_{(x, \omega) \in \mathbf{R} \times \Omega} \widetilde{M}_i(x, \omega) & \underline{M}^- &:= \max_{1 \leq i \leq \tilde{L}} \widetilde{\underline{M}}_i \end{aligned}$$

Case 1: $\min\{\underline{M}^+, \underline{M}^-\} > 0$. Denote

$$k_+ = \max\{1 \leq i \leq L | \underline{M}_i > 0\} \quad k_- = \max\{1 \leq i \leq \tilde{L} | \widetilde{\underline{M}}_i > 0\}$$

$$\widehat{H}(p, x, \omega) := \begin{cases} \mathcal{L}|p - \widetilde{q}_{k_-}| + H(\widetilde{q}_{k_-}, x, \omega) & p \in (-\infty, \widetilde{q}_{k_-}) \\ H(p, x, \omega) & p \in [\widetilde{q}_{k_-}, q_{k_+}] \\ \mathcal{L}|p - q_{k_+}| + H(q_{k_+}, x, \omega) & p \in (q_{k_+}, +\infty) \end{cases}$$

By section 8, $\widehat{H}(p, x, \omega)$ has a level-set convex effective Hamiltonian $\overline{\widehat{H}}(p) \geq 0$ with $\overline{\widehat{H}}(0) = 0$. For any $\lambda > 0$, let $v_\lambda(x, q, \omega)$ and $\widehat{v}_\lambda(x, q, \omega)$ be solutions of the following equations:

$$\lambda v_\lambda + H(q + v'_\lambda, x, \omega) = 0 \quad \lambda \widehat{v}_\lambda + \widehat{H}(q + \widehat{v}'_\lambda, x, \omega) = 0$$

Claim: $q_{k_-} < q < q_{k_+}$.

Proof of the Claim: Suppose it is not true.

(I) If $q = q_{k_+}$, then $0 = \overline{H}(q) = \overline{H}(q_{k_+}) \geq \underline{M}_{k_+} > 0$, this is a contradiction.

(II) If $q > q_{k_+}$. The arguments are divided into the following (II-1), (II-2) and (II-3).

(II-1) By Lemma 4.1, there is $\Omega_1 \subset \Omega$, $\mathbf{P}[\Omega_1] = 1$ such that if $\omega \in \Omega_1$, then for any $R > 0$, there is $\lambda_1 = \lambda_1(R, q, \omega) > 0$,

$$0 < \lambda < \lambda_1 \implies q + v'_\lambda(x, q, \omega) \geq q_{k_+} \quad x \in [-\frac{R}{\lambda}, \frac{R}{\lambda}]$$

(II-2) By (4.3), there are $\delta > 0$ and $\tau > 0$ such that $\mathbf{E}[m(x, \omega) > \delta] = \tau$. By ergodic theorem, there is $\Omega_2 \subset \Omega$, $\mathbf{P}[\Omega_2] = 1$. For each $\omega \in \Omega_2$ and $R > 0$,

$$\lim_{\lambda \rightarrow 0} \frac{2\lambda}{R} \int_{-\frac{R}{\lambda}}^{\frac{R}{\lambda}} \chi_{\{m(\cdot, \omega) > \delta\}}(x) dx = \mathbf{E}[m(x, \omega) > \delta] = \tau$$

So there is some $\lambda_2(R, q, \omega) > 0$, such that

$$0 < \lambda < \lambda_2(R, q, \omega) \implies \frac{2\lambda}{R} \int_{-\frac{R}{\lambda}}^{\frac{R}{\lambda}} \chi_{\{m(\cdot, \omega) > \delta\}}(x) dx > \frac{\tau}{2}$$

(II-3) Since $\overline{H}(q) = 0$, there is $\Omega_3 \subset \Omega$, $\mathbf{P}[\Omega_3] = 1$. For each $\omega \in \Omega_3$ and $R > 0$, there is $\lambda_3 = \lambda_3(R, q, \omega) > 0$,

$$0 < \lambda < \lambda_3 \implies |\lambda v_\lambda(x, q, \omega)| < \delta \quad x \in [-\frac{R}{\lambda}, \frac{R}{\lambda}]$$

Let $\widetilde{\lambda}(R, q, \omega) := \min\{\lambda_1(R, q, \omega), \lambda_2(R, q, \omega), \lambda_3(R, q, \omega)\} > 0$, $\widetilde{\Omega} := \Omega_1 \cap \Omega_2 \cap \Omega_3$, then $\mathbf{P}[\widetilde{\Omega}] = 1$, for each $\omega \in \widetilde{\Omega}$, when $\lambda < \widetilde{\lambda}(R, q, \omega)$, there is $x_\lambda \in [-\frac{R}{\lambda}, \frac{R}{\lambda}]$, $m(x_\lambda, \omega) > \delta$,

$$\delta < m(x_\lambda, \omega) \leq H(q + v'_\lambda(x_\lambda, q, \omega), x_\lambda, \omega) = -\lambda v_\lambda(x_\lambda, q, \omega) < \delta$$

This is a contradiction. (The second inequality is because we have $q + v'_\lambda(x_\lambda, q, \omega) \geq q_{k_+}$). So, $q < q_{k_+}$. Similarly, we can prove $q_{k_-} < q$. This ends the proof of the **Claim**.

By Lemma 4.1, there is $\widehat{\Omega}$, $\mathbf{P}[\widehat{\Omega}] = 1$. For $\omega \in \widehat{\Omega}$ and any $R > 0$, there is $\widehat{\lambda}(R, q, \omega) > 0$,

$$0 < \lambda < \widehat{\lambda} \implies q_{k_+} \leq q + v'_\lambda(x, q, \omega) \leq q_{k_+} \quad x \in [-\frac{R}{\lambda}, \frac{R}{\lambda}]$$

So

$$\lambda v_\lambda(x, q, \omega) + \widehat{H}(q + v'_\lambda(x, q, \omega), x, \omega) = 0 \quad x \in [-\frac{R}{\lambda}, \frac{R}{\lambda}]$$

By Lemma 2.9, there is some constant $C > 0$, such that

$$|\lambda v_\lambda(0, q, \omega) - \lambda \widehat{v}_\lambda(0, q, \omega)| \leq \frac{C}{R}$$

$R > 0$ can be chosen arbitrarily large, then

$$\overline{\widehat{H}}(q) = \lim_{\lambda \rightarrow 0} -\lambda \widehat{v}_\lambda(0, q, \omega) = \lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, q, \omega) = \overline{H}(q) = 0$$

By level-set convexity of $\widehat{H}(p)$ and $\widehat{H}(0) = 0$, $\widehat{H}|_{[0,q]} \equiv 0$. Since $\widehat{H}(p, x, \omega) \geq H(p, x, \omega)$, $\widehat{H}(p) \geq \overline{H}(p)$. On the other hand, $\overline{H}(p) \geq 0$. So $\overline{H}|_{[0,q]} \equiv 0$.

Case 2: $\min\{\underline{M}^+, \underline{M}^-\} \leq 0 < \max\{\underline{M}^+, \underline{M}^-\}$.

Construct $\widehat{H}(p, x, \omega)$ by modifying one side and similar arguments thereafter.

Case 3: $\max\{\underline{M}^+, \underline{M}^-\} \leq 0$.

By section 8, $\overline{H}(p)$ is level-set convex. Define $\widehat{H}(p, x, \omega) := H(p, x, \omega)$ and apply the result of **Case 1**. This ends the proof of (1).

The proof for (2) is similar. □

5. REDUCTION BY CONSTRAINED HAMILTONIAN WITH INDEX $(\tilde{L}, 0)$ AND $(0, L)$

Let $H(p, x, \omega)$ be a constrained Hamiltonian that satisfies **(A1)**-**(A3)**. Define

$$H^+(p, x, \omega) := \begin{cases} H(p, x, \omega) & p \geq 0 \\ \mathcal{L}|p| + H(0, x, \omega) & p < 0 \end{cases} \quad H^-(p, x, \omega) := \begin{cases} \mathcal{L}|p| + H(0, x, \omega) & p \geq 0 \\ H(p, x, \omega) & p < 0 \end{cases}$$

Lemma 5.1. *If both $H^+(p, x, \omega)$ and $H^-(p, x, \omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then $H(p, x, \omega)$ is also regularly homogenizable for all $p \in \mathbf{R}$ and*

$$\overline{H}(p) = \begin{cases} \overline{H}^+(p) & p \geq 0 \\ \overline{H}^-(p) & p < 0 \end{cases}$$

Proof. Fix $p \geq 0$, $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p, \omega)$ and $v_{+, \lambda}(x, p, \omega)$ be solutions of the equations

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad \lambda v_{+, \lambda} + H^+(p + v'_{+, \lambda}, x, \omega) = 0$$

By $H^+(p, x, \omega) \geq H(p, x, \omega)$, $\text{ess sup}_{(x, \omega)} H(p, x, \omega) \geq 0$ and comparison principle, we have

$$\liminf_{\lambda \rightarrow 0} -\lambda v_{+, \lambda}(0, p, \omega) \geq \liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) \geq 0$$

Thus, if $\overline{H}^+(p) = 0$, then $\overline{H}(p) = 0$. Since $\overline{H}^+(0) = 0$, we can only consider the case: $p > 0$ and $\overline{H}^+(p) > 0$. By Lemma 4.1, for a.e. $\omega \in \Omega$, any $R > 0$, there exists $\lambda_0 = \lambda_0(R, p, \omega) > 0$, such that

$$0 < \lambda < \lambda_0 \implies p + v_{+, \lambda}(x, \omega) \geq 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So $\lambda v_{+, \lambda} + H(p + v'_{+, \lambda}, x, \omega) = 0$, $\forall x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$. By Lemma 2.9, there is a constant $C > 0$ and

$$|\lambda v_{+, \lambda}(0, p, \omega) - \lambda v_\lambda(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large,

$$\lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0} -\lambda v_{+, \lambda}(0, p, \omega) = \overline{H}^+(p)$$

So $\overline{H}(p) = \overline{H}^+(p)$, $p \geq 0$. Similarly, we can prove $\overline{H}(p) = \overline{H}^-(p)$, $p \leq 0$. □

Remark 5.2. By Lemma 5.1, to prove the homogenization of a Hamiltonian that satisfies **(A1)**-**(A3)** and is constrained with index (\tilde{L}, L) , it suffices to study those Hamiltonians that have index $(0, L)$ or $(\tilde{L}, 0)$. Without loss of generality, in the following sections, we only consider the Hamiltonian under assumptions **(A1)**-**(A3)** and be constrained with index $(0, L)$.

6. GLUING LEMMAS: REDUCTION FROM SMALL OSCILLATION TO LARGE OSCILLATION

In this section, $H(p, x, \omega)$ satisfies **(A1)**-**(A3)** and is constrained with index $(0, L)$. Denote

$$\underline{M} := \operatorname{ess\,inf}_{(x, \omega) \in \mathbf{R} \times \Omega} M(x, \omega) \quad \overline{m} := \operatorname{ess\,sup}_{(x, \omega) \in \mathbf{R} \times \Omega} m(x, \omega)$$

There are $1 \leq \underline{k}, \overline{k} \leq L$, such that

$$\underline{M} := \operatorname{ess\,inf}_{(x, \omega) \in \mathbf{R} \times \Omega} M_{\underline{k}}(x, \omega) \quad \overline{m} := \operatorname{ess\,sup}_{(x, \omega) \in \mathbf{R} \times \Omega} m_{\overline{k}}(x, \omega)$$

Definition 6.1 (Oscillation). Let $H(p, x, \omega)$ be constrained(3.2) and satisfies **(A1)**-**(A3)**.

(1) $H(p, x, \omega)$ has small oscillation if $\underline{M} \geq \overline{m}$.

(2) $H(p, x, \omega)$ has large oscillation if $\underline{M} < \overline{m}$.

Throughout this section, we assume small oscillation and denote

$$P := p_{\overline{k}} \quad Q := q_{\underline{k}}$$

6.1. **Left Steep Side:** $\underline{M} > \overline{m}$ and $P < Q$. Define

$$\begin{aligned} H_1(p, x, \omega) &:= \begin{cases} H(p, x, \omega) & p \leq Q \\ \mathcal{L}|p - Q| + H(Q, x, \omega) & p > Q \end{cases} \\ H_3(p, x, \omega) &:= \begin{cases} H(p, x, \omega) & p \geq q_{\overline{k}} \\ \mathcal{L}|p - q_{\overline{k}}| + H(q_{\overline{k}}, x, \omega) & p < q_{\overline{k}} \end{cases} \\ H_2(p, x, \omega) &:= \max\{H_1(p, x, \omega), H_3(p, x, \omega)\} \end{aligned}$$

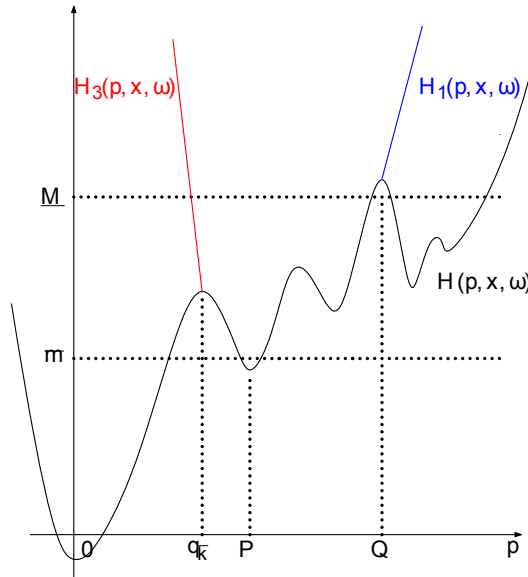


FIGURE 3. Left Steep Side

Lemma 6.2. Assume $H_i(p, x, \omega)$, $i = 1, 2, 3$ are all regularly homogenizable for any $p \in \mathbf{R}$. Then $H(p, x, \omega)$ is also regularly homogenizable for any $p \in \mathbf{R}$ and

$$\overline{H}(p) = \min\{\overline{H}_1(p), \overline{H}_3(p)\}$$

Proof of the periodic case. For any $p \in \mathbf{R}$, we have the cell problem

$$(6.1) \quad H(p + v'(x), x) = \overline{H}(p)$$

Proof by contradiction, if there are $x_1, x_2 \in [0, 1]$, such that $p + v'(x_1) > Q$ and $p + v'(x_2) < P$. Then $px + v(x) - Qx$ attains local maximum at some $y_1 \in (x_1, x_2 + 1)$ and $px + v(x) - Px$ attains local minimum at some $y_2 \in (x_2, x_1 + 1)$. Thus we get a contradiction from equalities:

$$\min_{x \in [0, 1]} H(Q, x) = \underline{M} \leq H(Q, y_1) \leq \overline{H}(p) \leq H(P, y_2) \leq \overline{m} = \max_{x \in [0, 1]} H(P, x)$$

Thus, either $p + v'(x) \leq Q$ for all $x \in [0, 1]$ or $p + v'(x) \geq P$ for all $x \in [0, 1]$. By (6.1), either $\overline{H}(p) = \overline{H}_1(p)$ or $\overline{H}(p) = \overline{H}_3(p)$. On the other hand, since $H(p, x, \omega) = \min\{H_1(p, x, \omega), H_3(p, x, \omega)\}$, by comparison principle, we have $\overline{H}(p) \leq \{\overline{H}_1(p), \overline{H}_3(p)\}$. Eventually, we conclude

$$\overline{H}(p) = \{\overline{H}_1(p), \overline{H}_3(p)\}$$

□

Proof of the random case. Decompose \mathbf{R} into three parts.

(1) If $p \in (-\infty, P)$, then $\overline{H}(p) = \overline{H}_1(p)$.

For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p, \omega)$ and $v_{1,\lambda}(x, p, \omega)$ be solutions of the equations

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad \lambda v_{1,\lambda} + H_1(p + v'_{1,\lambda}, x, \omega) = 0$$

By Lemma 4.2, there is $\tilde{\Omega} \subset \Omega$, $\mathbf{P}[\tilde{\Omega}] = 1$. For $\omega \in \tilde{\Omega}$, any $R > 0$, there is $\lambda_0 = \lambda_0(R, p, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p + v'_{1,\lambda}(x, p, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Thus, for $0 < \lambda < \lambda_0(R, p, \omega)$,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \lambda v_{1,\lambda} + H(p + v'_{1,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{1,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \rightarrow 0^+} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{1,\lambda}(0, p, \omega) = \overline{H}_1(p)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H}_1(p)$, $p \in (-\infty, P)$.

(2) $p \in (Q, \infty)$, then $\overline{H}(p) = \overline{H}_3(p)$.

For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p, \omega)$ and $v_{3,\lambda}(x, p, \omega)$ be solutions of the equations

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad \lambda v_{3,\lambda} + H_3(p + v'_{3,\lambda}, x, \omega) = 0$$

By Lemma 4.2, there is $\tilde{\Omega} \subset \Omega$, $\mathbf{P}[\tilde{\Omega}] = 1$. For $\omega \in \tilde{\Omega}$, any $R > 0$, there exists some $\lambda_0 = \lambda_0(R, \omega, p) > 0$,

$$0 < \lambda < \lambda_0 \implies p + v'_{3,\lambda}(x, p, \omega) \geq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Thus, for $0 < \lambda < \lambda_0(R, p, \omega)$,

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \lambda v_{3,\lambda} + H(p + v'_{3,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{3,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \rightarrow 0^+} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{3,\lambda}(0, p, \omega) = \overline{H}_3(p)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H}_3(p)$, $p \in (Q, \infty)$.

(3.1) Denote:

$$A := \{p \in (P, Q) \mid \overline{m} < \overline{H}_2(p) < \underline{M}\}$$

Fix any $p \in A$, for any $\lambda > 0$, let $v_\lambda(x, p, \omega)$, $v_{2,\lambda}(x, p, \omega)$ be solutions of the equations:

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad \lambda v_{2,\lambda} + H_2(p + v'_{2,\lambda}, x, \omega) = 0$$

By Lemma 4.1, for each $\omega \in \tilde{\Omega}$, any $R > 0$, there is $\lambda_0 = \lambda_0(R, p, \omega) > 0$, s.t.

$$0 < \lambda < \lambda_0 \implies P \leq p + v'_{2,\lambda}(x, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \lambda v_{2,\lambda} + H(p + v'_{2,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{2,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \rightarrow 0^+} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{2,\lambda}(0, p, \omega) = \overline{H}_2(p)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H}_2(p) \geq \{\overline{H}_1(p), \overline{H}_3(p)\}$. On the other hand $\overline{H}(p) \leq \min\{\overline{H}_1(p), \overline{H}_3(p)\}$. So $\overline{H}(p) = \overline{H}_1(p) = \overline{H}_2(p) = \overline{H}_3(p)$, $p \in A$.

(3.2) For $p \in \mathbf{R}$, if $\overline{H}_1(p) < \underline{M}$, then $\overline{H}(p) = \overline{H}_1(p)$.

The assumption $\overline{H}_1(p) < \underline{M}$ implies $p < Q$. By Lemma 4.1, for $\omega \in \tilde{\Omega}$, any $R > 0$, there is $\lambda_0 = \lambda_0(R, p, \omega) > 0$, such that

$$0 < \lambda < \lambda_0 \implies p + \lambda v_{1,\lambda}(x, p, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \lambda v_{1,\lambda} + H(p + v'_{1,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{1,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \rightarrow 0^+} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{1,\lambda}(0, p, \omega) = \overline{H}_1(p)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H}_1(p)$.

(3.3) For $p > P$, if $\overline{H}_3(p) > \overline{m}$, then $\overline{H}(p) = \overline{H}_3(p)$.

By Lemma 4.1, for each $\omega \in \tilde{\Omega}$, any $R > 0$, there is $\lambda_0 = \lambda_0(R, p, \omega) > 0$, such that

$$0 < \lambda < \lambda_0 \implies p + \lambda v_{3,\lambda}(x, p, \omega) \geq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, \lambda v_{3,\lambda} + H(p + v'_{3,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{3,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\lim_{\lambda \rightarrow 0^+} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{3,\lambda}(0, p, \omega) = \overline{H}_3(p)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p and $\overline{H}(p) = \overline{H}_3(p)$.

(3.4) For $p < Q$, if $\overline{H}_3(p) < \underline{M}$, then $\overline{H}_2(p) = \overline{H}_3(p) < \underline{M}$.

By Lemma 4.1, for each $\omega \in \tilde{\Omega}$, any $R > 0$, there is $\lambda_0 = \lambda_0(R, p, \omega) > 0$, such that

$$0 < \lambda < \lambda_0 \implies p + v'_{3,\lambda}(x, p, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Here, for any $\lambda > 0$, $v_{3,\lambda}$ is the solution of the equation

$$\lambda v_{3,\lambda} + H_3(p + v'_{3,\lambda}, x, \omega) = 0$$

However, by the above upper bound,

$$\lambda v_{3,\lambda} + H_2(p + v'_{3,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Suppose for any $\lambda > 0$, $v_{2,\lambda}(x, p, \omega)$ is the solution of the equation:

$$\lambda v_{2,\lambda} + H_2(p + v'_{2,\lambda}, x, \omega) = 0$$

By Lemma 2.9, there is $C = C(p)$, such that

$$|\lambda v_{2,\lambda}(0, p, \omega) - \lambda v_{3,\lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large

$$\overline{H}_2(p) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{2,\lambda}(0, p, \omega) = \lim_{\lambda \rightarrow 0^+} -\lambda v_{3,\lambda}(0, p, \omega) = \overline{H}_3(p)$$

Now, we discuss the homogenization of $\overline{H}(p)$ for $p \in [P, Q] \cap A^c$.

(I) If $p \in (P, Q)$ and $\overline{H}_2(p) \leq \overline{m}$, by the fact $\overline{m} < \underline{M}$ and

$$\max\{\overline{H}_1(p), \overline{H}_3(p)\} \leq \overline{H}_2(p)$$

we have $\overline{H}_1(p) < \underline{M}$, by **(3.2)**, $\overline{H}(p) = \overline{H}_1(p)$.

(II) If $p \in (P, Q)$ and $\overline{H}_2(p) \geq \underline{M}$, then by **(3.4)**, $\overline{H}_3(p) \geq \underline{M} > \overline{m}$. By **(3.3)**, $\overline{H}(p) = \overline{H}_3(p)$.

(III) By Corollary 2.8, we have

$$\overline{H}(P) = \overline{H}_1(P) \text{ and } \overline{H}(Q) = \overline{H}_3(Q)$$

In all, for any $p \in \mathbf{R}$, either $\overline{H}(p) = \overline{H}_1(p)$ or $\overline{H}(p) = \overline{H}_3(p)$, so

$$\overline{H}(p) \geq \min\{\overline{H}_1(p), \overline{H}_3(p)\}$$

On the other hand

$$\overline{H}(p) \leq \min\{\overline{H}_1(p), \overline{H}_3(p)\}$$

So, we have proved:

$$\overline{H}(p) = \min\{\overline{H}_1(p), \overline{H}_3(p)\}$$

□

6.2. **Right Steep Side:** $\underline{M} > \overline{m}$ and $Q \leq P$. Define

$$H_1(p, x, \omega) := \begin{cases} H(p, x, \omega) & p \leq Q \\ \mathcal{L}|p - Q| + H(Q, x, \omega) & p > Q \end{cases}$$

$$H_2(p, x, \omega) = \begin{cases} -\mathcal{L}|p| + H(0, x, \omega) & p < 0 \\ H(p, x, \omega) & 0 \leq p \leq P \\ -\mathcal{L}|p - P| + H(P, x, \omega) & p > P \end{cases}$$

$$H_3(p, x, \omega) := \begin{cases} H(p, x, \omega) & p \geq Q \\ \mathcal{L}|p - Q| + H(Q, x, \omega) & p < Q \end{cases}$$

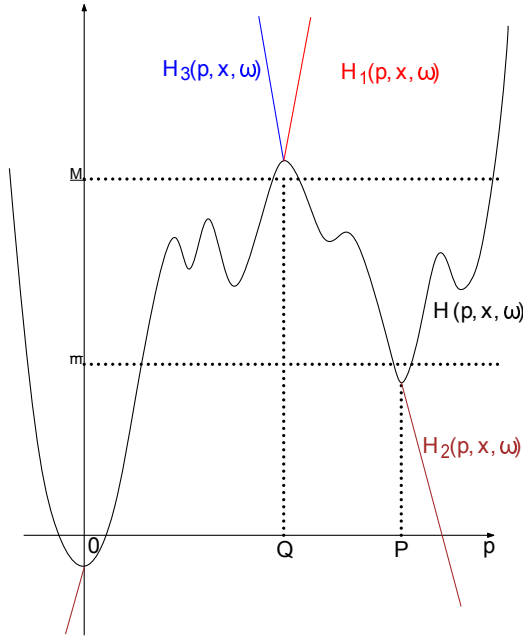


FIGURE 4. Right Steep Side

Lemma 6.3. Assume both $H_1(p, x, \omega)$ and $H_3(p, x, \omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then $H(p, x, \omega)$ is also regularly homogenizable for all p and

$$\overline{H}(p) = \begin{cases} \overline{H}_1(p) & p \leq 0 \\ \min\{\overline{H}_1(p), \overline{H}_3(p), \underline{M}\} & p \in (0, P) \\ \overline{H}_3(p) & p \geq P \end{cases}$$

Proof of the periodic case for middle equality. For $p \in (0, P)$, we have the cell problem

$$H(p + v'(x), x) = \overline{H}(p)$$

If $p + v'(x) \leq Q, \forall x \in [0, 1]$ or $p + v'(x) \geq Q, \forall x \in [0, 1]$, then $\overline{H}(p) = \overline{H}_1(p)$ or $\overline{H}(p) = \overline{H}_3(p)$. Otherwise, by the assumption that $\underline{M} > \overline{m}$, we have $p + v'(x) \in [0, P], \forall x \in [0, 1]$.

There is some $x_0 \in [0, 1]$, such that $H(Q, x_0) = \min_x \max_{q \in [0, P]} H(q, x) = \underline{M}$. So we have $\overline{H}(p) = H(p + v'(x_0), x_0) \leq \underline{M}$. Thus

$$\overline{H}(p) \leq \min\{\overline{H}_1(p), \overline{H}_3(p), \underline{M}\}$$

If $\overline{H}(p) < \underline{M}$, then by Lemma 4.1, we have either $p + v'(x) \leq Q, \forall x \in [0, 1]$ or $p + v'(x) \geq Q, \forall x \in [0, 1]$ and so $\overline{H}(p) = \overline{H}_1(p)$ or $\overline{H}(p) = \overline{H}_3(p)$. \square

Proof of random case. STEP 1: Proof of the first equality. Define

$$f(\theta) := \operatorname{ess\,sup}_{(x, \omega) \in \mathbf{R} \times \Omega} [H(\theta Q, x, \omega)]$$

Then $f(0) = 0, f(1) \geq \underline{M} > \overline{m} > 0$.

By the continuity of f , there is some $\theta_0 \in (0, 1)$, such that $0 < f(\theta_0) < \underline{M}$.

For any $p \leq 0, \lambda > 0$, let $v_{1, \lambda}(x, p, \omega)$ be the solution of the equation

$$\lambda v_{1, \lambda} + H_1(p + v'_{1, \lambda}, x, \omega) = 0$$

Apply Lemma 4.2 to $(p, \theta_0 Q, Q)$ and $H_1(p, x, \omega)$, then for a.e. $\omega \in \Omega$, we have: for any $R > 0$, there exists $\lambda_0 = \lambda_0(R, p, \omega) > 0$,

$$0 < \lambda < \lambda_0 \implies p + v'_{1, \lambda}(x, p, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Then by the definition of $H_1(p, x, \omega)$, we have

$$\lambda v_{1, \lambda} + H(p + v'_{1, \lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

For any $\lambda > 0$, let v_λ be the unique viscosity solution of the equation

$$\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0 \quad x \in \mathbf{R}$$

By Lemma 2.9, there is $C = C(p) > 0$, such that

$$|\lambda v_\lambda(0, p, \omega) - \lambda v_{1, \lambda}(0, p, \omega)| \leq \frac{C}{R}$$

Since R can be chosen arbitrarily large,

$$\lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) = \lim_{\lambda \rightarrow 0} -\lambda v_{1, \lambda}(0, p, \omega) = \overline{H}_1(p)$$

Thus, H is regularly homogenizable at p and

$$\overline{H}(p) = \overline{H}_1(p) \quad p \leq 0$$

STEP 2: Proof of the third equality. Similar as the proof of Step 1.

STEP 3: The second equality.

(3.1) Claim: For $p_0 \in \mathbf{R}$, if $\overline{H}_1(p_0) < \underline{M}$, then $H(p, x, \omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H}_1(p_0)$.

Proof of (3.1) Claim. By the definition of $H_1(p, x, \omega)$, $\overline{H}_1(p_0) < \underline{M}$ implies $p < Q$ (since $\overline{H}_1(p) \geq \underline{M}$ for $p \geq Q$). For each $\omega \in \Omega$ and $\lambda > 0$, let $v_\lambda(x, p_0, \omega)$ and $v_{1, \lambda}(x, p_0, \omega)$ be solutions of the equations

$$\lambda v_\lambda + H(p_0 + v'_\lambda, x, \omega) = 0 \quad \lambda v_{1, \lambda} + H_1(p_0 + v'_{1, \lambda}, x, \omega) = 0$$

By Lemma 4.1, for a.e. $\omega \in \Omega$, we have: for each $R > 0$, there is $\lambda_1 = \lambda_1(R, p_0, \omega) > 0$, such that

$$0 < \lambda < \lambda_1 \implies p_0 + v'_{1, \lambda} \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So

$$\lambda v_{1,\lambda} + H(p_0 + v'_{1,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is $C = C(p_0) > 0$, such that

$$|\lambda v_\lambda(0, p_0, \omega) - \lambda v_{1,\lambda}(0, p_0, \omega)| < \frac{C}{R}$$

Since we can choose arbitrary large R , we have that

$$\lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p_0, \omega) = \lim_{\lambda \rightarrow 0} -\lambda v_{1,\lambda}(0, p_0, \omega) = \overline{H}_1(p_0)$$

Thus $H(p, x, \omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H}_1(p_0)$. \square

(3.2) Claim: For $p_0 \in \mathbf{R}$, if $\overline{H}_3(p_0) < \underline{M}$, then $H(p, x, \omega)$ is regularly homogenizable at p_0 and $\overline{H}(p_0) = \overline{H}_3(p_0)$.

Proof of (3.2) Claim. The proof is similar as **(3.1) Claim**. \square

(3.3) Denote

$$q_1 = \min \{p \in [0, P] \mid \overline{H}_1(p) = \underline{M}\} \quad q_2 = \max \{p \in [0, P] \mid \overline{H}_3(p) = \underline{M}\}$$

(3.1), (3.2) \implies $H(p, x, \omega)$ is regularly homogenizable for $p \in (0, q_1) \cup (q_2, P)$ and

$$\overline{H}(p) = \begin{cases} \overline{H}_1(p) & p \in (0, q_1) \\ \overline{H}_3(p) & p \in (q_2, P) \end{cases}$$

By Corollary 2.8, $H(p, x, \omega)$ is regularly homogenizable at q_1 and q_2 and

$$\overline{H}(q_1) = \overline{H}(q_2) = \underline{M}$$

(3.4) Claim: $H_2(p, x, \omega)$ is regularly homogenizable at q_1 and q_2 , moreover,

$$\overline{H}_2(q_1) = \overline{H}_2(q_2) = \underline{M}$$

Proof of (3.4) Claim. By the definition, we have $q_1, q_2 \in (0, P_0)$. For any $\omega \in \Omega$, $\lambda > 0$, let $v_\lambda(x, q_i, \omega)$ and $v_{2,\lambda}(x, q_i, \omega)$ be solutions to the following equations

$$\lambda v_\lambda + H(q_i + v'_\lambda, x, \omega) = 0 \quad \lambda v_{2,\lambda} + H_2(q_i + v'_{2,\lambda}, x, \omega) = 0$$

By the fact that

$$\overline{H}(q_i) = \underline{M} > \max_{(x,\omega)} \{\text{ess sup } H(0, x, \omega), H(P, x, \omega)\} = \overline{m}$$

By Lemma 4.1, then for a.e. $\omega \in \Omega$, for any $R > 0$, there is $\lambda_2 = \lambda_2(q_i, R, \omega) > 0$,

$$0 < \lambda < \lambda_2 \implies 0 \leq q_i + v_\lambda(x, q_i, \omega) \leq Q \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

So we have

$$\lambda v_\lambda + H_2(q_i + v'_\lambda, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Apply Lemma 2.9, there is some constant $C = C(q_i) > 0$, such that

$$|\lambda v_\lambda(0, q_i, \omega) - \lambda v_{2,\lambda}(0, q_i, \omega)| < \frac{C}{R}$$

We can choose arbitrary large R , so

$$\lim_{\lambda \rightarrow 0} -\lambda v_{2,\lambda}(0, q_i, \omega) = \lim_{\lambda \rightarrow 0} -\lambda v_\lambda(0, q_i, \omega) = \overline{H}(q_i) = \underline{M}$$

Thus, $H_2(p, x, \omega)$ is regularly homogenizable at q_i and $\overline{H}_2(q_i) = \underline{M}$. \square

(3.5) Denote

$$\widehat{M}(x, \omega) := \max_{\substack{\bar{k} \leq j \leq L \\ j \neq \underline{k}}} M_j(x, \omega)$$

Then we have

$$\underline{\widehat{M}} := \operatorname{ess\,inf}_{(x, \omega) \in \mathbf{R} \times \Omega} \widehat{M}(x, \omega) \leq \underline{M}$$

By Lemma 2.6, without loss of generality, we can further assume $\underline{\widehat{M}} < \underline{M}$. This means that $\mathbf{E}[\widehat{M}(x, \omega) < \underline{M}] > 0$. Denote $\widetilde{H}(p, x, \omega) := -H_2(q_{\underline{k}, 0} - p, x, \omega) + \underline{M}$.

If $w_\lambda(x, p, \omega)$ is a viscosity solution to

$$\lambda w_\lambda + H_2(p + w'_\lambda, x, \omega) = 0$$

Then $\widetilde{w}_\lambda(x, p, \omega) := -w_\lambda(x, p, \omega)$ is a viscosity solution to

$$\lambda \widetilde{w}_\lambda + \widetilde{H}(q_{\underline{k}, 0} - p + \widetilde{w}'_\lambda, x, \omega) + \underline{M} = 0$$

Apply Lemma 4.3 to $\widetilde{H}(p, x, \omega)$, we deduce that $\overline{H_2}|_{[q_1, q_2]} \equiv \underline{M}$.

(3.6) For each $p \in [q_1, q_2]$, let $v_\lambda(x, p, \omega)$ be the solution to

$$\lambda v_\lambda(x, p, \omega) + H(p + v'_\lambda(x, p, \omega), x, \omega) = 0$$

By that fact that $H(p, x, \omega) \geq H_2(p, x, \omega)$, we have

$$\mathbf{E}[\omega \in \Omega \mid \liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) \geq \underline{M}] = 1$$

We only need to show

$$\mathbf{E}[\omega \in \Omega \mid \limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) \leq \underline{M}] = 1$$

(3.7) Define $\widehat{H}_2(p, x, \omega)$ as following:

$$\widehat{H}_2(p, x, \omega) = \begin{cases} H_2(p, x, \omega) & p \in (-\infty, 0) \cup (P, \infty) \\ \text{concave envelope of } H_2(p, x, \omega)|_{p \in [0, P]} & p \in [0, P] \end{cases}$$

By definition, $\widehat{H}_2(p, x, \omega)$ is determined by those stationary functions: $m_i(x, \omega), M_j(x, \omega)$, $1 \leq i, j \leq L$, so $\widehat{H}_2(p, x, \omega)$ is stationary. Then by the theory of level-set convex homogenization, $\widehat{H}_2(p, x, \omega)$ can be homogenized to some level-set concave effective Hamiltonian $\overline{\widehat{H}_2}(p) \leq \underline{M}$.

Since $\widehat{H}_2(p, x, \omega) \geq H_2(p, x, \omega)$, there exists $\widehat{q}_1 < \widehat{q}_2$ such that $[q_1, q_2] \subset [\widehat{q}_1, \widehat{q}_2]$ and $\overline{\widehat{H}_2}(\widehat{q}_1) = \overline{\widehat{H}_2}(\widehat{q}_2) = \underline{M}$. By level-set concavity, $\overline{\widehat{H}_2}|_{[\widehat{q}_1, \widehat{q}_2]} = \underline{M}$.

Denote

$$\widetilde{H}_2(p, x, \omega) = \min\{\widehat{H}_2(p, x, \omega), \underline{M}\}$$

Then $\widetilde{H}_2(p, x, \omega)$ has a level-set concave effective Hamiltonian $\overline{\widetilde{H}_2}(p)$ with

$$\overline{\widetilde{H}_2}|_{[\widehat{q}_1, \widehat{q}_2]} = \underline{M}$$

For any $p_1 \in [\widehat{q}_1, \widehat{q}_2]$ and $\lambda > 0$, let $\widehat{v}_\lambda(x, p_1, \omega)$ be the solution of the equation

$$\lambda \widehat{v}_{2, \lambda} + \widehat{H}_2(p_1 + \widehat{v}'_{2, \lambda}, x, \omega) = 0$$

We will have

$$\lim_{\lambda \rightarrow 0} \inf_{x \in B_R} -\lambda \widehat{v}_{2, \lambda}(x, p_1, \omega) \geq \underline{M}$$

Since $p_1 < P$ and $0 < \overline{m} < \underline{M}$, by Lemma 4.1, we have that: for a.e. $\omega \in \Omega$, any $R > 0$, there is some $\lambda_0 = \lambda_0(R, p_1, \omega) > 0$, when $\lambda < \lambda_0$,

$$0 \leq p_1 + \widehat{v}'_{2,\lambda}(x, p_1, \omega) \leq P \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

Define

$$\widehat{H}(p, x, \omega) = \begin{cases} H(p, x, \omega) & p \in (-\infty, 0) \cup (P, \infty) \\ \widehat{H}_2(p, x, \omega) & p \in [0, P] \end{cases}$$

For each $\omega \in \Omega$ and $\lambda > 0$, let $\widehat{v}_\lambda(x, p_1, \omega)$ be the solution of the equation

$$\lambda \widehat{v}_\lambda + \widehat{H}(p_1 + \widehat{v}'_\lambda, x, \omega) = 0$$

Thus

$$\lambda \widehat{v}_{2,\lambda} + \widehat{H}(p_1 + \widehat{v}'_{2,\lambda}, x, \omega) = 0 \quad x \in \left[-\frac{R}{\lambda}, \frac{R}{\lambda}\right]$$

By Lemma 2.9, there is some constant $C = C(p_1) > 0$, such that

$$\left| \lambda \widehat{v}_\lambda(0, p_1, \omega) - \lambda \widehat{v}_{2,\lambda}(0, p_1, \omega) \right| \leq \frac{C}{R}$$

We can choose arbitrary large R , so

$$\lim_{\lambda \rightarrow 0} -\lambda \widehat{v}_\lambda(0, p_1, \omega) = \lim_{\lambda \rightarrow 0} -\lambda \widehat{v}_{2,\lambda}(0, p_1, \omega) = \underline{M}$$

This means that

$$\overline{\widehat{H}}|_{[q_1, q_2]} \equiv \underline{M}$$

By the fact that $\widehat{H}(p, x, \omega) \geq H(p, x, \omega)$, we have

$$\mathbf{E}[\omega \in \Omega | \limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p, \omega) \leq \underline{M}] = 1$$

This completes the proof. □

Lemma 6.4. *Let $H(p, x, \omega)$ be constrained Hamiltonian that satisfies **(A1)-(A3)** and $\underline{M} = \overline{m}$, then there is a family of Hamiltonians $\{H_n(p, x, \omega)\}_{n \in \mathbf{N}}$, each $H_n(p, x, \omega)$ is a constrained Hamiltonian and satisfies **(A1)-(A3)**, moreover, we have $\underline{M}_n > \overline{m}_n$ and*

$$\|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(\mathbf{R} \times \mathbf{R} \times \Omega)} \leq \frac{1}{n}$$

Proof. For each $n \in \mathbf{N}$, define the function

$$h_n(p, x, \omega) := \begin{cases} \frac{p - q_k}{n(p_k^- - q_k)} & p \in [q_k, p_k^-] \\ -\frac{p - p_k^-}{n(q_{k-1}^- - p_k^-)} + \frac{1}{n} & p \in (p_k^-, q_{k-1}^-) \\ 0 & \text{elsewhere} \end{cases}$$

And define

$$H_n(p, x, \omega) := H(p, x, \omega) - h_n(p, x, \omega)$$

Since $q_k(x, \omega)$, $p_k^-(x, \omega)$ and $q_{k-1}^-(x, \omega)$ are all stationary, $H_n(p, x, \omega)$ is also stationary. By the construction, we have

$$\overline{m}_n = \overline{m} - \frac{1}{n} = \underline{M} - \frac{1}{n} < \underline{M}_n - \frac{1}{n} < \underline{M}_n$$

Moreover

$$\|H_n(p, x, \omega) - H(p, x, \omega)\|_{L^\infty(\mathbf{R} \times \mathbf{R} \times \Omega)} = \frac{1}{n}$$

□

Remark 6.5. In Lemma 6.4, if those $H_n(p, x, \omega)$ are regularly homogenizable for all $p \in \mathbf{R}$, then according to Lemma 2.6, $H(p, x, \omega)$ is also regularly homogenizable and $\overline{H}(p) = \lim_{n \rightarrow \infty} \overline{H}_n(p)$.

Remark 6.6. The point of Lemma 6.2, Lemma 6.3 and Lemma 6.4 is: to prove the homogenization of constrained Hamiltonian $H(p, x, \omega)$ with index $(0, L)$ and has small oscillation, it suffices to study the homogenization of constrained Hamiltonian $H(p, x, \omega)$ with index $(0, L)$ and has large oscillation.

7. AUXILIARY LEMMAS FOR LARGE OSCILLATION

7.1. Existence Lemma.

Lemma 7.1. *Let Hamiltonian $H(p, x, \omega)$ satisfy (A1)-(A3) and be constrained with index $(0, L)$, then for any $\mu \geq 0, \omega \in \Omega$, there is a Lipschitz continuous viscosity solution $u(x, \omega)$ to the equation:*

$$\begin{cases} H(u', x, \omega) = \mu \\ u' \geq 0 \end{cases} \quad x \in \mathbf{R}$$

Proof. Fix $\mu \geq 0$ and $\omega \in \Omega$. By (A2), there exists $p_0 > 0$, such that $H(p_0, x, \omega) > \mu$. Since $H(0, x, \omega) \leq \mu$, $u_+ := p_0 x$ is a super-solution and $u_- := C$ is a sub-solution for any constant C .

STEP 1. Fix $a \in \mathbf{R}$ and let $C_a := p_0 a$, then

$$u_+(a, \omega) = u_-(a, \omega) \text{ and } u_+(x, \omega) > u_-(x, \omega) \quad \forall x \in (a, \infty)$$

Define

$$u_a(x, \omega) := \sup_v \{v(x, \omega) \in C([a, \infty)) | H(v', x, \omega) \leq \mu, C_a \leq v(x, \omega) \leq p_0 x\}$$

Then

$$\begin{cases} H(u'_a, x, \omega) = \mu & x \in (a, \infty) \\ u_a(a, \omega) = p_0 a \end{cases}$$

STEP 2. Fix any $a < b$, denote

$$w(x, \omega) := u_a(x, \omega) + [u_b(b, \omega) - u_a(b, \omega)] \quad x \geq b$$

Then

$$\begin{cases} H(w', x, \omega) = \mu & x \in (b, \infty) \\ w(b, \omega) = p_0 b \end{cases}$$

So $u_b(x, \omega) \geq w(x, \omega)$ on $[b, \infty)$. Denote

$$\tilde{u}_a(x, \omega) := \begin{cases} u_a(x, \omega) & x \in [a, b] \\ u_b(x, \omega) - u_b(b, \omega) + u_a(b, \omega) & x \in (b, \infty) \end{cases}$$

Then

$$p_0 x \geq \tilde{u}_a(x, \omega) \geq u_a(x, \omega) \geq C_a, \quad x \in [a, \infty)$$

On the other hand, by the construction, $\tilde{u}_a(x, \omega)$ is a sub-solution, so $\tilde{u}_a(x, \omega) \leq u_a(x, \omega)$. Thus $\tilde{u}_a(x, \omega) \equiv u_a(x, \omega)$, which means

$$(u_b(x, \omega) - u_a(x, \omega))|_{(b, \infty)} \equiv u_b(b, \omega) - u_a(b, \omega)$$

The above equality is true for any $a < b$, this also implies $u'_a(x, \omega) \geq 0$.

STEP 3. For any $n \in \mathbf{Z}$, then

$$u_n(x, \omega) - u_n(0, \omega) = u_{n+1}(x, \omega) - u_{n+1}(0, \omega) \quad \forall x \geq n + 1$$

For any $x \in \mathbf{R}$ let $m := [x]$ and define

$$u(x, \omega) := u_m(x, \omega) - u_m(0, \omega)$$

So $u(x, \omega)$ is a well defined Lipschitz function on \mathbf{R} and it is the solution of the equation

$$\begin{cases} H(u', x, \omega) = \mu \\ u' \geq 0 \end{cases} \quad x \in \mathbf{R}$$

□

7.2. Decomposition Lemma.

Lemma 7.2. *Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and be constrained with index $(0, L)$. Let u be a Lipschitz continuous viscosity solution of the equation*

$$\begin{cases} H(u'(x, \omega), x, \omega) = \mu \geq 0 \\ u'(x, \omega) \geq 0 \end{cases} \quad x \in \mathbf{R}$$

Then there is a sequence $\{b_i\}_{i \in \mathbf{Z}}$, such that

$$\lim_{i \rightarrow \pm\infty} b_i = \pm\infty, u \in C^1(I_i), I_i = (b_i, b_{i+1})$$

$$u'(x, \omega)|_{I_i} = \psi_{k_i, (x, \omega)}(\mu) \text{ for some } k_i \in \{1, 2, \dots, 2L + 1\}$$

Proof. Fix $\omega \in \Omega$ and omit the notation ω .

STEP 1. **Claim:** for each $x \in \mathbf{R}$, there exist $\delta_x > 0$ and $l_x, r_x \in \{1, 2, \dots, 2L + 1\}$, such that

$$u'(y) = \begin{cases} \psi_{l_x, y}(\mu) & y \in (x - \delta_x, x) \\ \psi_{r_x, y}(\mu) & y \in (x, x + \delta_x) \end{cases}$$

Just prove the first equality, since the proof for the second one is similar. Suppose this is not true at some x_0 , then there exist two sequences $x_n \rightarrow x_0^-$ and $y_n \rightarrow x_0^-$, $1 \leq k_2 < k_1 \leq 2L + 1$, such that

$$x_1 < y_1 < x_2 < y_2 < \dots < x_0 \quad u'(x_n) = \psi_{k_1, x_n}(\mu) \quad u'(y_n) = \psi_{k_2, y_n}(\mu)$$

Case 1: $k_1 \geq k_2 + 2$. Then there is a branch between the k_1 -th branch and the k_2 -th branch. So there exist $a < b$, such that $u'(x_n) < a < b < u'(y_n)$.

Fix any $p \in [a, b]$, then $u(x) - px$ is decreasing(increasing) around $x_n(y_n)$. So, $u(x)$ attains local minimum(maximum) at $z_n^- \in (x_n, y_n)(z_n^+ \in (y_n, x_{n+1}))$, then $H(p, z_n^+) \leq \mu \leq H(p, z_n^-)$, and thus there is $z_n \in [z_n^-, z_n^+]$ with $H(p, z_n) = \mu$. By the fact $\lim_{n \rightarrow \infty} z_n = x_0$, we have $H(p, x_0) = \mu$.

This is true for any $p \in [a, b]$ and this contradicts to the fact that $H(p, x, \omega)$ is constrained.

Case 2: $k_1 = k_2 + 1$, without loss of generality, let $k_1 = 2, k_2 = 1$.

If $m_1(x_0) < \mu$, by the similar argument used in **Case 1**, we get a contradiction.

If $m_1(x_0) > \mu$, there is some $\delta > 0$, s.t. $m_1(\cdot)|_{(x_0-\delta, x_0)} > \mu$, let $x_n \in (x_0 - \delta, x_0)$, then $\mu = H(u'(x_n), x_n) \geq H(p_1, x_n) > \mu$, which is a contradiction.

If $m_1(x_0) = \mu$, since $m_1(x)$ has no cluster point, there is some $\delta > 0$ such that $\mu \notin \{m_1(x)|x \in (x_0 - \delta, x_0)\}$. By the above discussion, $m_1(\cdot)|_{(x_0-\delta, x_0)} < \mu$. Let $\Phi(x) := u(x) - p_1x$, then $\Phi'(x_n) < 0$ and $\Phi'(y_n) > 0$, so there is some $z_n \in (x_n, y_n)$ where $\Phi(x)$ attains local minimum. So $m_1(z_n) = H(p_1, z_n) \geq \mu$, since $z_n \in (x_0 - \delta, x_0)$ when $n \gg 1$, we get the contradiction.

Thus, the **Claim** is proved.

STEP 2. Denote: $A := \{x \in \mathbf{R} | l_x \neq r_x\}$. By the above arguments, we see that A has no cluster point. Then there is a sequence $\{b_i\}_{i \in \mathbf{Z}}$ such that $b_i < b_{i+1}$, $A \subset \{b_i\}_{i \in \mathbf{Z}}$ and $\lim_{i \rightarrow \pm\infty} b_i = \pm\infty$. We will have $r_{b_i} = l_{b_{i+1}}$. Thus $u'(x) = \psi_{r_{b_i}, x}(\mu)$, $x \in (b_i, b_{i+1})$

□

7.3. Homotopy between solutions. Let $H(p, x, \omega)$ be constrained with index $(0, L)$, for simplicity of notation, we omit the dependence of ω . Let $f \in L^\infty(\mathbf{R})$ and any solution of $u'(x) = f(x)$ is a viscosity solution to

$$\begin{cases} H(u', x) = f(x) \\ u' \geq 0 \end{cases} \quad x \in \mathbf{R}$$

By Lemma 7.2, let $a_1 < a_2 < a_3$ and $f(x)|_{(a_i, a_{i+1})} = \psi_{k_i, x}(\mu)$, $k_i \in \{1, 2, \dots, 2L+1\}$, $i = 1, 2$. Denote $k = \min\{k_1, k_2\}$ and define

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \mathbf{R} \setminus (a_1, a_3) \\ \psi_{k, x}(\mu) & x \in (a_1, a_3) \end{cases}$$

Lemma 7.3. Assume $\mu \notin \{m_i(x), M_j(x) | 1 \leq i, j \leq L, x \in (a_1, a_3)\}$. Then any solution of $u' = \tilde{f}$ is also a viscosity solution of

$$H(u'(x), x) = \mu \quad x \in \mathbf{R}$$

Proof. Similar to the proof of A.3 in [4].

□

Let $I = (a, b)$, and $f_1, f_2 \in L^\infty(I)$, $f_1 \geq f_2$. Assume solutions of

$$\begin{cases} u'_1 = f_1 & x \in I \\ u_1(a) = 0 \end{cases} \quad \text{and} \quad \begin{cases} u'_2 = f_2 & x \in I \\ u_2(a) = 0 \end{cases}$$

are both viscosity solutions of the equation

$$(7.1) \quad H(u', x, \omega) = \mu \quad x \in I$$

Then $u_2(x) \leq u_1(x) \leq u_2(x) - u_2(b) + u_1(b)$. Fix any $c \in [u_2(b), u_1(b)]$ and define

$$\begin{aligned} u_{c,*}(x) &:= \max\{u_2(x), u_1(x) - u_1(b) + c\} \\ u^{c,*}(x) &:= \min\{u_1(x), u_2(x) - u_2(b) + c\} \end{aligned}$$

Define the set

$$\mathcal{W} := \{w \in W^{1,\infty}(I) | H(w', x, \omega) \leq \mu \text{ and } u_{c,*}(x) \leq w(x) \leq u^{c,*}(x)\}$$

And the function $w_c(x) := \sup_{w \in \mathcal{W}} w(x)$. Denote

$$\mathcal{F}_I(f_1, f_2, c)(x) := \begin{cases} w'_c(x) & \text{if } w_c \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

Then $u_{c,*}(x)(u^{c,*}(x))$ is a viscosity sub(super)solution to equation (7.1). By Perron's method, $w_c(x)$ is a viscosity solution of the equation

$$\begin{cases} H(w'_c(x), x) = \mu \\ w_c(a) = 0, w_c(b) = c \end{cases} \quad x \in (a, b)$$

Lemma 7.4. Fix $a < b$, $0 < \epsilon < \frac{b-a}{2}$, let $f_1, f_2 \in L^\infty(a - \epsilon, b + \epsilon)$ such that

$$f_1(x) \geq f_2(x), \quad x \in (a - \epsilon, b + \epsilon) \quad f_1(x) = f_2(x), \quad x \in (a - \epsilon, a) \cup (b, b + \epsilon)$$

Suppose any solution of $\begin{cases} u'_i(x) = f_i(x) & x \in (a - \epsilon, b + \epsilon) \\ u_i(a) = 0 \end{cases}$ ($i = 1, 2$) is a viscosity (sub-)solution of the equation: $H(u', x) = \mu$. Fix $c \in [u_2(b), u_1(b)]$, then any solution of the equation

$$v'(x) = \begin{cases} f_1(x) = f_2(x) & x \in (a - \epsilon, a) \cup (b, b + \epsilon) \\ \mathcal{F}_I(f_1, f_2, c)(x) & x \in I = (a, b) \end{cases}$$

is a viscosity (sub-)solution of the equation

$$\begin{cases} H(u'(x), x) = \mu & x \in (a - \epsilon, b + \epsilon) \\ u(b) = c \end{cases}$$

Proof. Same as the proof of Lemma A.4 in [4]. □

8. HOMOGENIZATION OF HAMILTONIAN WITH LARGE OSCILLATION

In this section, the Hamiltonian is assumed to satisfy **(A1)**-(**A3**), be constrained(c.f. Definition 3.2) with index $(0, L)$ and has large oscillation(c.f. Definition 6.1).

8.1. Admissible decomposition and admissible functions. Recall 3.1, 3.2 and denote

$$\underline{m} := \operatorname{ess\,inf}_{(x, \omega)} m(x, \omega) \quad \overline{M} := \operatorname{ess\,sup}_{(x, \omega)} M(x, \omega) \quad \mathcal{P} = (\underline{m}, \overline{M}) \cap [0, \infty)$$

Definition 8.1. Fix any $\mu \in \mathcal{P}$ and $\omega \in \Omega$, a collection of disjoint finite intervals $\{I_i\}_{i \in \mathbf{Z}}$ is called a (μ, ω) admissible decomposition of \mathbf{R} if the following (1),(2) and (3) hold.

- (1) $I_i = (a_i, a_{i+1})$, $\bigcup_{i \in \mathbf{Z}} [a_i, a_{i+1}] = \mathbf{R}$
- (2) $\mu \in \{m_j(a_i, \omega), M_j(a_i, \omega), m_j(a_i, \omega), M_j(a_i, \omega) | 1 \leq j \leq L\}$.
- (3) $\mu \notin \{m_j(x, \omega), M_j(x, \omega) | 1 \leq j \leq L, x \in (a_i, b_i)\}$.

Remark 8.2. Since $H(p, x, \omega)$ is constrained and has large oscillation, such $\{I_i\}_{i \in \mathbf{Z}}$ exists and is unique. By **(A1)**, for any $y \in \mathbf{R}$, $\{I_i - y\}_{i \in \mathbf{Z}}$ is the $(\mu, \tau_y \omega)$ admissible decomposition of \mathbf{R} .

Definition 8.3. For fixed $\omega \in \Omega$ and $\mu \in \mathcal{P}$, let $\{I_i\}_{i \in \mathbf{Z}}$ be a (μ, ω) admissible decomposition of \mathbf{R} , then $f : \mathbf{R} \rightarrow \mathbf{R}$ is a (μ, ω) admissible function if following (1),(2) and (3) hold.

- (1) $0 \leq f(x) \leq \max\{p \geq 0 | H(p, x, \omega) \leq \overline{M}\}$.
- (2) For each $i \in \mathbf{Z}$, $f(x)|_{I_i} = \psi_{j_i, x}(\mu)$, for some $j_i \in \{1, 2, \dots, 2L + 1\}$.
- (3) Any solution of $u' = f(x)$ is a viscosity solution of the equation

$$(8.1) \quad \begin{cases} H(u'(x), x, \omega) = \mu \\ u' \geq 0 \end{cases} \quad x \in \mathbf{R}$$

Definition 8.4. For $\mu \geq 0$ and $\omega \in \Omega$, define

$$\mathcal{A}_\mu(\omega) := \begin{cases} \{\text{All } (\mu, \omega) \text{ admissible functions} \} & \mu \in \mathcal{P} \\ \psi_{2L+1,x}(\mu) & \mu \leq \underline{m} \geq 0 \\ \psi_{1,x}(\mu) & \mu \geq \overline{M} \end{cases}$$

Lemma 8.5. $\mathcal{A}_\mu(\omega) \neq \emptyset$.

Proof. Fix $\omega \in \Omega$, by Lemma 7.1, there is a viscosity solution $u(x)$ of the equation (8.1).

By Lemma 7.2, there is a strictly increasing sequence $\{b_i\}_{i \in \mathbf{Z}}$ such that

$$\lim_{i \rightarrow \pm\infty} b_i = \pm\infty; \quad u \in C^1((b_i, b_{i+1})), i \in \mathbf{Z}; \quad u'(x)|_{(b_i, b_{i+1})} = \psi_{k_i, x}(\mu), k_i \in \{1, 2, \dots, 2L+1\}$$

Let $\mu \in \mathcal{P}$ and $\{I_j\}_{j \in \mathbf{Z}}$ be the (μ, ω) admissible decomposition of \mathbf{R} . By refinement, we may assume that for $i \in \mathbf{Z}$, $(b_i, b_{i+1}) \subset I_{l_i}$, $l_i \in \mathbf{Z}$. For each $j \in \mathbf{Z}$, denote: $s(j) = \min\{k_i | (b_i, b_{i+1}) \subset I_j\}$. And define $f(x, \omega) := \psi_{s(j), x}(\mu)$, $x \in I_j = (a_j, a_{j+1})$. By Lemma 7.3, any solution to $u' = f$ is a viscosity solution of the equation (8.1).

Thus $f \in \mathcal{A}_\mu(\omega)$. If $\mu \notin \mathcal{P}$, it is clear that $\mathcal{A}_\mu(\omega) \neq \emptyset$. □

Definition 8.6. For each $\omega \in \Omega$ and $\mu \geq 0$, denote

$$\bar{f}_\mu(x, \omega) = \sup\{f(x) | f \in \mathcal{A}_\mu(\omega)\} \quad \underline{f}_\mu(x, \omega) = \inf\{f(x) | f \in \mathcal{A}_\mu(\omega)\}$$

Lemma 8.7. (1) For any $\mu \geq 0$ and $\omega \in \Omega$, $\bar{f}_\mu(x, \omega), \underline{f}_\mu(x, \omega) \in \mathcal{A}_\mu(\omega)$.

(2) $\bar{f}_\mu(x, \omega) \geq \underline{f}_\mu(x, \omega)$ and both of them are stationary.

Proof. (1) Fix any $\mu \geq 0$ and $\omega \in \Omega$. For any point $x_0 \in \mathbf{R}$, since $H(p, x, \omega)$ is constrained with index $(0, L)$, there are $f_r \in \mathcal{A}_\mu(\omega)$, $\delta_r > 0$ and $k_r \in \{1, 2, \dots, 2L+1\}$, such that

$$\bar{f}_\mu(x, \omega)|_{(x_0, x_0 + \delta_r)} = f_r(x)|_{(x_0, x_0 + \delta_r)} = \psi_{k_r, x}(\mu)$$

Similarly, there are $f_l \in \mathcal{A}_\mu(\omega)$, $\delta_l > 0$ and $k_l \in \{1, 2, \dots, 2L+1\}$, such that

$$\bar{f}_\mu(x, \omega)|_{(x_0 - \delta_l, x_0)} = f_l(x)|_{(x_0 - \delta_l, x_0)} = \psi_{k_l, x}(\mu)$$

(i) If $k_l = k_r = k$. Then $\psi_{k, x}(\mu)$ is continuous on $(x_0 - \delta_l, x_0 + \delta_r)$. Since $H(\psi_{k, x}(\mu), x, \omega) = \mu$, any solution of $u' = \psi_{k, x}(\mu)$ is the solution of the equation: $H(u', x, \omega) = \mu$, $x \in (x_0 - \delta_l, x_0 + \delta_r)$

(ii) If $k_l < k_r$. It suffices to check any solution to $u' = \bar{f}_\mu$ is a viscosity sub-solution at x_0 . This follows from the following fact: $[\bar{f}(x_0^+), \bar{f}(x_0^-)] = [f_r(x_0^+), f_l(x_0^-)] \subset [f_l(x_0^+), f_l(x_0^-)]$.

(iii) If $k_l > k_r$. It suffices to check any solution to $u' = \bar{f}_\mu$ is a viscosity super-solution at x_0 . This follows from the following fact: $[\bar{f}(x_0^-), \bar{f}(x_0^+)] = [f_l(x_0^-), f_r(x_0^+)] \subset [f_r(x_0^-), f_r(x_0^+)]$.

So $\bar{f}_\mu(x, \omega) \in \mathcal{A}_\mu(\omega)$. Similarly, $\underline{f}_\mu(x, \omega) \in \mathcal{A}_\mu(\omega)$.

(2) By definition, $\bar{f}_\mu(\cdot, \omega) \geq \underline{f}_\mu(\cdot, \omega)$. By Remark 8.2, for any $y \in \mathbf{R}$,

$$\bar{f}(x, \tau_y \omega) = \sup\{f(x) | f(x) \in \mathcal{A}_\mu(\tau_y \omega)\} = \sup\{f(x) | f(x - y) \in \mathcal{A}_\mu(\omega)\} = \bar{f}(x + y, \omega)$$

Similarly, $\underline{f}(x, \tau_y \omega) = \underline{f}(x + y, \omega)$ for any $y \in \mathbf{R}$. □

8.2. Intermediate level set of the effective Hamiltonian.

Lemma 8.8. *Let $H(p, x, \omega)$ satisfy (A1)-(A3) and be constrained with index $(0, L)$. If $\mu > \underline{M}$, then for a.e. $\omega \in \Omega$, the following is true: for any $f(x) \in \mathcal{A}_\mu(\omega)$, there is a sequence of intervals $\{J_k\}_{k \in \mathbf{Z}}$ such that*

$$J_k = (c_k, c_{k+1}), \bigcup_{k \in \mathbf{Z}} [c_k, c_{k+1}] = \mathbf{R}, \lim_{k \rightarrow \pm\infty} c_k = \pm\infty, f|_{J_{2k}} = \psi_{1,(x,\omega)}(\mu)$$

Proof. By Lemma 2.5, for a.e. $\omega \in \Omega$, $\underline{M} = \text{ess inf}_{x \in \mathbf{R}} M(x, \omega)$. Denote $\delta := \mu - \underline{M}$ and $\epsilon := \frac{\delta}{2}$. By ergodic theorem,

$$\lim_{L \rightarrow \pm\infty} \frac{1}{L} \int_0^L \chi_{\{z, M(z, \omega) < \underline{M} + \epsilon\}}(x) dx = \mathbf{E}[M(0, \cdot) < \underline{M} + \epsilon] > 0 \quad \text{a.e. } \omega \in \Omega$$

So, almost surely, there is a sequence $x_i = x_i(\omega)$, such that $\lim_{i \rightarrow \pm\infty} x_i = \pm\infty$, $M(x_i, \omega) < \underline{M} + \epsilon$. By continuity of $M(x, \omega)$ in x , for each i , there is $\delta_i > 0$, such that: $M(x, \omega) < \underline{M} + \epsilon$, $x \in (x_i - \delta_i, x_i + \delta_i)$

Denote $c_{2k} := x_k - \delta_k$, $c_{2k+1} := x_k + \delta_k$, $J_k := (c_k, c_{k+1})$. Then $f(x)|_{J_{2k}} = \psi_{1,(x,\omega)}(\mu)$ follows from the fact that: $H(f(x), x, \omega) = \mu > \underline{M} + \epsilon > M(x, \omega)|_{J_{2k}}$, a.e. $\omega \in \Omega$. \square

Lemma 8.9. *Let $H(p, x, \omega)$ satisfy (A1)-(A3) and be constrained with index $(0, L)$. If $0 \leq \mu < \overline{m}$, then for a.e. $\omega \in \Omega$, the following is true: for any $f(x) \in \mathcal{A}_\mu(\omega)$, there is a sequence of intervals $\{J_k\}_{k \in \mathbf{Z}}$ such that*

$$J_k = (c_k, c_{k+1}), \bigcup_{k \in \mathbf{Z}} [c_k, c_{k+1}] = \mathbf{R}, \lim_{k \rightarrow \pm\infty} c_k = \pm\infty, f|_{J_{2k}} = \psi_{2L+1,x}(\mu)$$

Proof. Similar argument as Lemma 8.8. \square

Lemma 8.10. *Let $H(p, x, \omega)$ satisfy (A1)-(A3) and be constrained with index $(0, L)$. Fix any $\mu \geq 0$ and $p \in [\int_\Omega \underline{f}_\mu(0, \omega) d\omega, \int_\Omega \overline{f}_\mu(0, \omega) d\omega]$, there is a stationary function $f(x, \omega) : \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ such that*

$$(1) p = \int_\Omega f(0, \omega) d\omega.$$

$$(2) \text{For a.e. } \omega \in \Omega, \text{ any solution to } u' = f(x, \omega) \text{ is a solution of the equation: } H(u', x, \omega) = \mu.$$

Proof. Suppose $\underline{u}'(x, \omega) = \underline{f}_\mu(x, \omega)$ and $\overline{u}'(x, \omega) = \overline{f}_\mu(x, \omega)$, Lemma 8.7 implies $H(\underline{u}', x, \omega) = \mu$, $H(\overline{u}', x, \omega) = \mu$. Fix $\omega \in \Omega$, according to Lemma 8.8 and Lemma 8.9, there exists a sequence of intervals $\{I_k\}_{k \in \mathbf{Z}}$, $I_k = (a_k, a_{k+1})$, such that $\lim_{k \rightarrow \pm\infty} a_k = \pm\infty$ and

$$\underline{f}_\mu(x, \omega) = \overline{f}_\mu(x, \omega), x \in I_{2k} \quad \underline{f}_\mu(x, \omega) \leq \overline{f}_\mu(x, \omega), x \in I_{2k+1}$$

Denote

$$\underline{d}_i = \int_{a_i}^{a_{i+1}} \underline{f}_\mu(s, \omega) ds \quad \overline{d}_i = \int_{a_i}^{a_{i+1}} \overline{f}_\mu(s, \omega) ds$$

For each $t \in [0, 1]$, define $f_t : \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ by

$$f_t(x, \omega) := \begin{cases} \underline{f}_\mu(x, \omega) = \overline{f}_\mu(x, \omega) & x \in I_{2i} \\ \mathcal{F}_{I_{2i+1}}(\overline{f}_\mu, \underline{f}_\mu, t\overline{d}_i + (1-t)\underline{d}_i) & x \in I_{2i+1} \end{cases}$$

So $f_t(x, \omega)$ is stationary and

$$\int_{a_0}^{a_i} f_t(x, \omega) dx = \int_{a_0}^{a_i} t \bar{f}_\mu(x, \omega) + (1-t) \underline{f}_\mu(x, \omega) dx$$

By **(A2)**, \underline{f}_μ and \bar{f}_μ are bounded. Then there is some constant $C > 0$, such that

$$\frac{1}{|a_i - a_0|} \left| \int_{a_0}^{a_i} f_t(x, \omega) dx - \int_{a_0}^{a_i} f_s(x, \omega) dx \right| = |t - s| \left| \int_{a_0}^{a_i} (\bar{f}_\mu(s, \omega) - \underline{f}_\mu(s, \omega)) ds \right| \leq C |t - s|$$

Thus

$$\lim_{L \rightarrow \infty} \frac{1}{L} \left| \int_0^L f_t(x, \omega) dx - \int_0^L f_s(x, \omega) dx \right| \leq C |t - s|$$

By ergodic theorem, for a.e. $\omega \in \Omega$,

$$\lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L f_t(x, \omega) dx = \mathbf{E}[f_t(0, \omega)] \quad \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L f_s(x, \omega) dx = \mathbf{E}[f_s(0, \omega)]$$

Then $|\mathbf{E}[f_t(0, \omega)] - \mathbf{E}[f_s(0, \omega)]| \leq C |t - s|$. So $\mathbf{E}[f_t(0, \omega)]$ is a continuous function of t , thus

$$\bigcup_{t \in [0, 1]} \mathbf{E}[f_t(0, \omega)] = \left[\int_{\Omega} \underline{f}_\mu(0, \omega) d\omega, \int_{\Omega} \bar{f}_\mu(0, \omega) d\omega \right]$$

So for any $p \in [\int_{\Omega} \underline{f}_\mu(0, \omega) d\omega, \int_{\Omega} \bar{f}_\mu(0, \omega) d\omega]$, there is $t = t(p) \in [0, 1]$, s.t. $\mathbf{E}[f_t(0, \omega)] = p$.

Let u be the solution of $u' = f_t(x, \omega)$. By Lemma 7.4, u is a solution of $H(u', x, \omega) = \mu$. □

Lemma 8.11. *Let $H(p, x, \omega)$ satisfy **(A1)**-**(A3)** and be constrained with index $(0, L)$. Fix $\omega \in \Omega$, assume $\mu_m \rightarrow \mu$ and $f_m(x) \in \mathcal{A}_{\mu_m}(\omega)$. Then we have the following hold.*

(1) *If $\mu \in \mathcal{P}$, then $\limsup_{m \rightarrow \infty} f_m(x) \in \mathcal{A}_\mu(\omega)$ and $\liminf_{m \rightarrow \infty} f_m(x) \in \mathcal{A}_\mu(\omega)$.*

(2) *If $\underline{\mu} \geq 0$ and $\mu \leq \underline{\mu}$, then except a countable set,*

$$\limsup_{m \rightarrow \infty} f_m(x) = \liminf_{m \rightarrow \infty} f_m(x) = \psi_{2L+1, (x, \omega)}(\mu)$$

(3) *If $\mu \geq \overline{M}$, then except a countable set,*

$$\limsup_{m \rightarrow \infty} f_m(x) = \liminf_{m \rightarrow \infty} f_m(x) = \psi_{1, (x, \omega)}(\mu)$$

Proof. Only prove $f(x) = \limsup_{m \rightarrow \infty} f_m(x) \in \mathcal{A}_\mu(\omega)$. The proof for \liminf is similar.

(1) Let $\{I_i\}_{i \in \mathbf{Z}}$ be the (μ, ω) admissible decomposition of \mathbf{R} . Fix $k \in \mathbf{Z}$ and $\epsilon \ll 1$, there is $N \in \mathbf{N}$, when $m > N$, $\mu_m \notin \{m_i(x, \omega), M_j(x, \omega) | 1 \leq i, j \leq L, x \in (a_k + \epsilon, a_{k+1} - \epsilon) \cup (a_{k+1} + \epsilon, a_{k+2} - \epsilon)\}$.

There are $l, \tilde{l}, q, \tilde{q} \in \{1, 2, \dots, 2L+1\}$, $\{f_{l_n}\}_{n \geq 1}$ and $\{f_{q_n}\}_{n \geq 1}$, such that

$$f_{l_n}(x) = \begin{cases} \psi_{l, (x, \omega)}(\mu) & x \in (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_{\tilde{l}, (x, \omega)}(\mu) & x \in (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}) \end{cases} \quad f_{q_n}(x) = \begin{cases} \psi_{\tilde{q}, (x, \omega)}(\mu) & x \in (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_{q, (x, \omega)}(\mu) & x \in (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}) \end{cases}$$

$$f(x)|_{I_k} = \psi_{l, (x, \omega)}(\mu) \quad f(x)|_{I_{k+1}} = \psi_{q, (x, \omega)}(\mu)$$

It suffices to show that the solution of $u' = f$ is a viscosity solution of (8.1) at a_{k+1} .

Define $u_l \in W^{1,\infty}(a_k, a_{k+2})$ and $u_q \in W^{1,\infty}(a_k, a_{k+2})$ by solutions of

$$u'_l(x) = \begin{cases} \psi_{l,(x,\omega)}(\mu) & x \in I_k \\ \psi_{\tilde{l},(x,\omega)}(\mu) & x \in I_{k+1} \end{cases} \quad u'_q(x) = \begin{cases} \psi_{\tilde{q},(x,\omega)}(\mu) & x \in I_k \\ \psi_{q,(x,\omega)}(\mu) & x \in I_{k+1} \end{cases}$$

By stability of viscosity solutions, u_l and u_q are both viscosity solutions to

$$H(v'(x), x, \omega) = \mu \quad x \in (a_k, a_{k+2})$$

The jump of f at a_{k+1} is contained in the jump of u'_l or the jump of u'_q at a_{k+1} , so the solution of $u' = f$ is a viscosity solution of (8.1).

(2) Denote $A = \{x \in \mathbf{R} | \mu = m_i(x) \text{ for some } 1 \leq i \leq L\}$. Since each of $m_i(x, \omega)$ has no cluster point, A is countable. Since $\underline{m} \geq 0$ and $\mu \leq \underline{m}$, if $x \notin A$, then $\limsup_{m \rightarrow \infty} f_m(x) = \liminf_{m \rightarrow \infty} f_m(x) = \psi_{2L+1,(x,\omega)}(\mu)$.

(3) Denote $B = \{x \in \mathbf{R} | \mu = M_j(x) \text{ for some } 1 \leq j \leq L\}$. Since each $M_j(x, \omega)$ has no cluster point, B is countable. Since $\mu \geq \overline{M}$, if $x \notin B$, then $\limsup_{m \rightarrow \infty} f_m(x) = \liminf_{m \rightarrow \infty} f_m(x) = \psi_{1,(x,\omega)}(\mu)$. \square

Notation 8.12. For each $\mu \geq 0$, denote $\mathcal{I}_\mu = \left[\int_\Omega \underline{f}_\mu(0, \omega) d\omega, \int_\Omega \overline{f}_\mu(0, \omega) d\omega \right]$.

Remark 8.13. If $\mu \neq \nu$, then $\mathcal{I}_\mu \cap \mathcal{I}_\nu = \emptyset$.

Lemma 8.14. If $\lim_{m \rightarrow \infty} \mu_m = \mu$, then

$$\int_\Omega \overline{f}_\mu(0, \omega) d\omega \geq \limsup_{m \rightarrow \infty} \int_\Omega \overline{f}_{\mu_m}(0, \omega) d\omega \quad \int_\Omega \underline{f}_\mu(0, \omega) d\omega \leq \liminf_{m \rightarrow \infty} \int_\Omega \underline{f}_{\mu_m}(0, \omega) d\omega$$

Moreover, $\bigcup_{\mu \geq 0} \mathcal{I}_\mu = [q_0, \infty)$ with $q_0 = \int_\Omega \underline{f}_0(0, \omega) d\omega$.

Proof. Same as the proof of Lemma 3.8 in 4. \square

Denote $z_l(x, \omega) := \min \{p \leq 0 : H(q, x, \omega) \leq 0 \text{ on } [p, 0]\}$.

8.3. Extreme level set of effective Hamiltonian.

Lemma 8.15. Let $H(p, x, \omega)$ satisfy **(A1)**-(**A3**) and be constrained with index $(0, L)$. For any $p \in [\mathbf{E}[z_l(0, \omega)], \mathbf{E}[\underline{f}_0(0, \omega)]]$, there is a stationary function $f(x, \omega)$ such that $p = \mathbf{E}[f(0, \omega)]$ and any solution to $u' = f$ is a viscosity sub-solution of $H(u', x, \omega) = 0, x \in \mathbf{R}$.

Proof. Since $H(p, x, \omega)$ is constrained with index $(0, L)$, $\overline{m} = \operatorname{ess\,sup}_{(x,\omega) \in \mathbf{R} \times \Omega} m(x, \omega) > 0$. And by similar arguments in Lemma 8.8, then: for a.e. $\omega \in \Omega$, there is $\{b_i\}_{i \in \mathbf{Z}}$ such that

$$\lim_{i \rightarrow \pm\infty} b_i = \pm\infty, \quad m(x, \omega)|_{(b_{2i}, b_{2i+1})} \in \left[\frac{3\overline{m}}{4}, \overline{m}\right], \quad m(x, \omega)|_{(b_{2i+1}, b_{2i+2})} \leq \frac{3}{4}\overline{m}$$

For each $i \in \mathbf{Z}$, denote $\underline{r}_i = \int_{b_i}^{b_{i+1}} z_l(x, \omega) d\omega$ and $\overline{r}_i = \int_{b_i}^{b_{i+1}} \underline{f}_0(x, \omega) dx$. Fix $t \in (0, 1)$, define a stationary function $f_t(x, \omega) : \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ by the following procedure.

STEP 1: Modification on (b_{2i}, b_{2i+1}) . Denote

$$f_{l,t}(x, \omega) = \begin{cases} (1-t)\underline{f}_0(x, \omega) + tz_l(x, \omega) & x \in \bigcup_{i \in \mathbf{Z}} (b_{2i}, b_{2i+1}) \\ z_l(x, \omega) & x \in \bigcup_{i \in \mathbf{Z}} [b_{2i+1}, b_{2i+2}] \end{cases}$$

$$f_{r,t}(x, \omega) = \begin{cases} (1-t)\underline{f}_0(x, \omega) + tz_l(x, \omega) & x \in \bigcup_{i \in \mathbf{Z}} (b_{2i}, b_{2i+1}) \\ \underline{f}_0(x, \omega) & x \in \bigcup_{i \in \mathbf{Z}} [b_{2i+1}, b_{2i+2}] \end{cases}$$

Since $H(p, x, \omega)$ is convex in p on $(z_l(x, \omega), \underline{f}_0(x, \omega))$ for all $x \in (b_{2i}, b_{2i+1})$, if u is the a solution of the equation $u' = f_{l,t}$ or $u' = f_{r,t}$, then in viscosity sense, we have $H(u'(x, \omega), x, \omega) \leq 0, x \in \mathbf{R}$

STEP 2: Modification on $[b_{2i+1}, b_{2i+2}]$. Define

$$f_t := \begin{cases} \mathcal{F}_{I_{2i+1}}(\underline{f}_0, z_l(x, \omega), (1-t)\bar{r}_i + t\underline{r}_i) & x \in [b_{2i+1}, b_{2i+2}] \\ f_{l,t}(x, \omega) = f_{r,t}(x, \omega) & x \in (b_{2i}, b_{2i+1}) \end{cases}$$

By Lemma 7.4, if $u' = f_t$, then in viscosity sense, we have $H(u'(x, \omega), x, \omega) \leq 0, x \in \mathbf{R}$.

By similar arguments as Lemma 8.10, there is some constant $C > 0$, such that

$$\begin{aligned} \frac{1}{|b_i - b_0|} \left| \int_{b_0}^{b_i} f_t(x, \omega) dx - \int_{b_0}^{b_i} f_s(x, \omega) dx \right| &\leq C|t - s| \\ \lim_{L \rightarrow +\infty} \frac{1}{L} \left| \int_0^L f_t(x, \omega) dx - \int_0^L f_s(x, \omega) dx \right| &\leq C|t - s| \end{aligned}$$

So $\mathbf{E}[f_t(0, \omega)]$ is a continuous function. Since $\mathbf{E}[f_0(0, \omega)] = \mathbf{E}[\underline{f}_0(0, \omega)]$, $\mathbf{E}[f_1(0, \omega)] = \mathbf{E}[z_l(0, \omega)]$, it concludes that $\bigcup_{t \in [0,1]} \mathbf{E}[f_t(0, \omega)] = [\mathbf{E}[z_l(0, \omega)], \mathbf{E}[\underline{f}_0(0, \omega)]]$. So for any $p \in [\mathbf{E}[z_l(0, \omega)], \mathbf{E}[\underline{f}_0(0, \omega)]]$, there is $t = t(p)$, such that $p = \mathbf{E}[f_t(0, \omega)]$, then any solution of $u' = f_t(x, \omega)$ is a viscosity sub-solution of $H(v', x, \omega) = 0, x \in \mathbf{R}$. □

Lemma 8.16. *Let $H(p, x, \omega)$ satisfy (A1)-(A3) and be constrained with index $(0, L)$. Then for a.e. $\omega \in \Omega$, we have: fix $p \in \mathbf{R}$, let $v_\lambda(\cdot, \omega) \in W^{1,\infty}(\mathbf{R})$ be the unique viscosity solution of the equation: $\lambda v_\lambda + H(p + v'_\lambda, x, \omega) = 0, x \in \mathbf{R}$, then $\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(x, \omega) \geq 0$.*

Proof. By assumption, $\operatorname{ess\,inf}_{(x,\omega) \in \mathbf{R} \times \Omega} H(0, x, \omega) < 0$, for each $(x, \omega) \in \mathbf{R} \times \Omega$, denote

$$V(x, \omega) := \min\{H(0, x, \omega), m(x, \omega)\}$$

Then $V(x, \omega) \leq 0$ and it is a bounded continuous stationary function. Then

$$H_+(p, x, \omega) := H(p, x, \omega) - V(x, \omega) \geq 0$$

For a.e. $\omega \in \Omega$, by similar argument as Lemma 8.8: for any $\delta > 0$, there are

$$I_i = (a_i, a_{i+1}) \quad \lim_{i \rightarrow \pm\infty} a_i = \pm\infty \quad -\delta \leq V(x, \omega) \leq 0, \quad x \in (a_{2i}, a_{2i+1})$$

Then $\liminf_{\substack{\lambda \rightarrow 0 \\ x \in (a_{2i}, a_{2i+1})}} -\lambda v_\lambda(x, \omega) \geq -\delta$.

On the other hand, for each $\omega \in \Omega$, there is a sequence $\lambda_n \rightarrow 0$ and a constant $C \in \mathbf{R}$, such that

$$-\lambda_n v_{\lambda_n}(x, \omega) \rightarrow C \quad \text{locally uniformly in } \mathbf{R}$$

So $C \geq -\delta$. Since $\delta > 0$ can be arbitrary, $C \geq 0$. Thus $\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(x, \omega) \geq 0$. □

Remark 8.17. By Lemma 8.15 and Lemma 8.16, for any $p \in [\mathbf{E}[z_l(0, \omega)], \mathbf{E}[\underline{f}_0(0, \omega)]]$, $H(p, x, \omega)$ is regularly homogenizable and $\overline{H}(p) = 0$.

Lemma 8.18. *For $p \in (-\infty, \mathbf{E}[z_l(0, \omega)])$, $H(p, x, \omega)$ is regularly homogenizable.*

Proof. For each $\mu \geq 0$, denote $p_\mu = \mathbf{E}[\Psi_{(0, \omega)}(\mu)]$, let $v(x, \omega)$ be the solution of the equation

$$v'(x, \omega) = \Psi_{(x, \omega)}(\mu) - p_\mu$$

Then v is a sub-linear solution of $H(p + v', x, \omega) = \mu, x \in \mathbf{R}$. The lemma follows from the fact that

$$(-\infty, \mathbf{E}[z_l(0, \omega)]) = \bigcup_{\mu > 0} \{p_\mu\}$$

□

Remark 8.19. From the construction of the effective Hamiltonian $\overline{H}(p)$, in the case of large oscillation, $\overline{H}(p)$ is coercive, continuous and level-set convex.

Acknowledgement: The author would like to thank his advisor Yifeng Yu for his helpful guidance and generous support.

REFERENCES

1. S.N.Armstrong and P.Cardaliaguet, Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions, preprint, arXiv:1504.02045 [math.AP].
2. S.N.Armstrong and P.E.Souganidis, *Stochastic homogenization of level-set convex Hamilton-Jacobi equations*,
3. S.N.Armstrong, H.V.Tran and Y.Yu . Stochastic homogenization of a nonconvex Hamilton-Jacobi equation, preprint, arXiv:1311.2029[math.AP].
4. S.N.Armstrong, H.V.Tran and Y.Yu. Stochastic homogenization of nonconvex Hamilton-Jacobi equations in one space dimension, preprint, arXiv:1410.7053 [math.AP].
5. A. Davini and A. Siconolfi. Exact and approximate correctors for stochastic Hamiltonians: the 1-dimensional case. Math. Ann., 345(4):749782, 2009.
6. W.Jing, P.E.Souganidis and H.V.Tran. Large time average of reachable sets and Applications to Homogenization of interfaces moving with oscillatory spatio-temporal velocity, preprint, arXiv:1408.2013v1 [math.AP].
7. F.Rezakhanlou and J.E.Tarver. Homogenization for stochastic Hamilton-Jacobi equations. Arch.Ration.Mech. Anal. **151**(2000), no.4,277-309.
8. R.W.Schwab. Stochastic homogenization of Hamilton-Jacobi equations in stationary ergodic spatio-temporal media. Indiana Univ. Math.J. 58, **2**(2009), 537-581.
9. P.Souganidis. Stochastic homogenization of Hamilton-Jacobi equations and some applications. Asymptot. Anal. **20**(1999), no.1,1-11.

UNIVERSITY OF CALIFORNIA, IRVINE, CA, 92697, USA
E-mail address: hongweig@math.uci.edu